# WORKING PAPER

## BETA-ADJUSTED COVARIANCE ESTIMATION

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February 2021
2021/1010

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# Beta-Adjusted Covariance Estimation

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**Abstract**

The increase in trading frequency of Exchanged Traded Funds (ETFs) presents a positive externality for financial risk management when the price of the ETF is available at a higher frequency than the price of the component stocks. The positive spillover consists in improving the accuracy of pre-estimators of the integrated covariance of the stocks included in the ETF. The proposed Beta Adjusted Covariance (BAC) equals the pre-estimator plus a minimal adjustment matrix such that the covariance-implied stock-ETF beta equals a target beta. We focus on the Hayashi and Yoshida (2005) pre-estimator and derive the asymptotic distribution of its implied stock-ETF beta. The simulation study confirms that the accuracy gains are substantial in all cases considered. In the empirical part of the paper, we show the gains in tracking error efficiency when using the BAC adjustment for constructing portfolios that replicate a broad index using a subset of stocks.

*Keywords:* High-frequency data; realized covariances; ETF; asynchronicity; stock-ETF beta; Localized Hayashi-Yoshida; Index tracking.

JEL: C22, C51, C53, C58

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This research has benefited from the financial support of the Flemish Science Foundation (FWO). We are grateful to Dries Cornilly, Olivier Scaillet and Tim Verdonck for their constructive comments. We also thank participants at various conferences and seminars for helpful comments.

# 1 Introduction

Accurate estimation of the covariation between asset returns is indispensable in various areas in finance such as asset pricing, portfolio optimization, and risk management (Duffie and Pan (1997); Jagannathan and Ma (2003)). The advent of common availability of high-frequency asset price data has spurred the development of methods for the ex post estimation of the covariation over a fixed time interval such as a trading day. A seminal contribution in this field is Barndorff-Nielsen and Shephard (2004) introducing the asymptotic distribution theory of realized covariance estimators. When using ultra high-frequency transaction data, the standard realized covariance estimator may no longer be a reliable estimator of the covariation of asset returns due to the non-synchronous trading of assets and microstructure noise. An abundant literature in financial econometrics has addressed this issue by developing alternative methods for computing realized covariances using the vector of high-frequency stock prices as input (see, e.g., Aït-Sahalia et al. (2010), Christensen et al. (2010), Hayashi and Yoshida (2005), Aït-Sahalia et al. (2010), Zhang (2011), Mancini and Gobbi (2012), Boudt et al. (2017) and Bollerslev et al. (2020), among others).

We make further progress by including exchange traded funds (ETFs) price information when estimating the realized covariance of the assets included in the ETF. The rationale for doing so is that, for popular ETFs like the SPY and XLF tracking the (financial firms in the) S&P 500, high-frequency ETF prices are today observed at a higher frequency than the stock prices of most of the ETF components. The joint observation of the ETF price and the price of one stock thus carries information about the return covariation of that stock and all other stocks. To capture that information, we study a new integrated quantity called the stock-ETF beta, defined as the continuous part of the quadratic covariation between the efficient price of a given stock and the weighted average efficient price of all stocks included in the ETF.

We describe three ways to estimate the stock-ETF beta. The first one is the covariance-implied stock-ETF beta. It is an integrated version of functions of the spot covariance matrix estimate associated to a realized covariance matrix estimator of the price vector of all stocks included in the ETF. The second one is the pairwise realized stock-ETF beta corresponding to an estimate obtained using the synchronized series of stock prices and ETF prices. The third one is to use expert opinion regarding the stock-ETF beta for each asset. Due to difficulties of estimating a covariance matrix using high-frequency prices, we expect that the latter two approaches are more accurate. This insight leads us to develop an estimation framework aiming at improving the initial realized covariance estimator based on the observed difference between its implied stock-ETF beta and a target stock-ETF beta obtained using pairwise estimation or expert opinion. The proposed framework is called Beta Adjusted Covariance (BAC) estimation.

Under the BAC framework, we refer to the realized covariance computed from stock prices only as the pre-estimator, while the pairwise or expert opinion based stock-ETF beta estimate is the target beta. The latter is the oracle beta when it is free of estimation error. We propose to adjust the pre-estimator such that its implied stock-ETF beta equals the target beta under the criterion of minimizing the distance between the adjusted estimator and the pre-estimator.

The pre-estimator used in our analysis is the one proposed by Hayashi and Yoshida (2005) which remains consistent and unbiased when using high-frequency prices of transactions of assets occurring asynchronously. We refer to it as the HY estimator and use its localized version (see e.g. Christensen et al. (2013)) to estimate the stock-ETF beta. Our choice is motivated by the efficiency results as obtained in Jacod and Rosenbaum (2013), and later extended by Li et al. (2019). However, the results obtained in the latter references cannot be applied to our situation mainly because our parameter of interest is a random transformation of the spot volatility. Thus, to our knowledge, the asymptotic distribution results presented in this paper are new within the theory of estimation of volatility functionals. We also propose a modification to the HY and BAC estimator such that they remains accurate estimators of the integrated covariance in the presence of price jumps and microstructure noise.

We conduct a Monte Carlo study to evaluate the accuracy gains (in terms of mean squared error) when the pre-estimator is the traditional realized covariance, the two-time scale estimator proposed by Zhang (2011) or the Hayashi and Yoshida (2005) estimator. We find that the accuracy gains are over 50% in the case in which the oracle beta is used as target, and remain economically significant when the target beta is estimated using the ETF and the stock log-price series.

We apply the BAC estimator to the Trades and Quotes millisecond transaction data of stocks included in the S&P 500 Financial Select Sector SPDR Fund with ticker XLF. Our sample runs from Jan 1, 2018 to Dec 31, 2019. For our sample, only seven out of the around 67 XLF components have a higher number of observations than the ETF. We study the performance gains of the BAC adjustment to the pre-estimator for constructing index tracking portfolios for the XLF using different subsets of its components. We find that, for the vast majority of cases, the next day's realized tracking error is lower when the portfolio is optimized using the BAC adjusted estimator than when using the HY pre-estimator itself.

## 2 Notation, model and the pre-estimator

### 2.1 Model and parameters of interest

We consider $d$ assets. Their underlying $d$-dimensional process of log-prices $X_t$ is defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and is supposed to be a continuous Itô's semimartingale, i.e.,

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s, \;\; t \geq 0, \tag{1}$$

in which $\mu$ is predictable and càdlàg (or càglàd) and $B$ is a $d'$-dimensional standard Brownian motion. We will also use the notation $\Sigma = \sigma\sigma'$ for the spot covariation, and we write

$$A_t = \int_0^t \mu_s ds, \;\; M_t = \int_0^t \sigma_s dB_s, \;\; t \geq 0. \tag{2}$$

Additionally, $\sigma$ is assumed to be a $d \times d'$ matrix-valued Itô's semimartingale, i.e., for $k = 1, \dots, d$, $l = 1, \dots, d'$, it holds that

$$\begin{aligned}\sigma_t^{kl} =& \sigma_0^{kl} + \int_0^t \tilde{\mu}_s^{kl} ds + \sum_{m=1}^{d'} \int_0^t \tilde{\sigma}_s^{klm} dB_s^m \\ &+ \int_0^t \int_E \varphi^{kl}(s,z) \mathbf{1}_{\|\varphi(s,z)\| \leq 1} (N - \lambda)(dsdz) + \int_0^t \int_E \varphi^{kl}(s,z) \mathbf{1}_{\|\varphi(s,z)\| > 1} N(dsdz),\end{aligned} \tag{3}$$

where $\tilde{\mu}$ and $\tilde{\sigma}$ are predictable and càdlàg (or càglàd), $\varphi$ is predictable and $N$ is a Poisson random measure with compensator $\lambda(dsdz) = ds\nu(dz)$ for some $\sigma$-finite measure $\nu$ defined in a Polish space $E$. Moreover, there is a localizing sequence $(\tau_n)_{n \geq 1}$ of stopping times as well as a deterministic sequence of non-negative functions $\Gamma_n$, such that $\int_E \Gamma_n(z)^2 \nu(dz) < \infty$, and for all $(\omega, t, z)$

$$\|\varphi(\omega, t, z)\| \wedge 1 \leq \Gamma_n(z), \;\; \text{whenever } t \leq \tau_n(\omega).$$

For every $l = 1, \dots, d$, we denote the set of $n_l$ observation times of the $l$-th log-price contained in $X_t$ by

$$\mathbb{T}^l = \left\{ 0 = t_0^l < \cdots < t_{n_l}^l \leq 1 \right\}. \tag{4}$$

Within this framework, we write

$$n = \sum_{k=1}^{d} n_k. \tag{5}$$

The object of interest is the integrated covariance matrix of the process $X_t$ over the interval $[0, 1]$:

$$\Theta = \int_0^1 \Sigma_s ds. \tag{6}$$

In order to estimate $\Theta$, we consider an ETF that is invested in each of those $d$ assets with the following time-varying amounts invested per share of the ETF:

$$a_t = (a_t^1, \dots, a_t^d)'. \tag{7}$$

The process $a_t$ is assumed to be a càdlàg step function. The corresponding log-transformed Net Asset Value

(NAV) is equal to the natural logarithm of the weighted sum of the component prices of the ETF:

$$Y_t^* = \log\left(\sum_{k=1}^{d} a_t^k \exp(X_t^k)\right). \tag{8}$$

Throughout the paper, we define the stock-ETF beta associated to the $l$-th asset, further denoted as $\beta^l$, as the continuous part of the quadratic covariation between $X^l$ and $\exp(Y^*)$. It follows from Itô's lemma that $\beta^l$ equals

$$\beta^l = \sum_{k=1}^{d} \int_0^1 w_s^k \Sigma_s^{kl} ds, \tag{9}$$

where

$$w_s^l = a_s^l \exp(X_s^l). \tag{10}$$

### 2.2 The pre-estimator and its implied stock-ETF beta

For every $0 \leq t \leq 1$, we denote by $\overline{\Sigma_t}$ a pre-estimator for the integrated covariance matrix $\int_0^t \Sigma_s ds$.* Based on this pre-estimator, we now introduce the implied estimators $\hat{\Sigma}_s^{kl}$ and $\overline{\beta}^l$ for the spot covariation $\Sigma_s$ and the stock-ETF beta $\beta^l$, respectively. We use a local estimation window of $k_n \in \mathbb{N}$ observations in order to define the following pre-estimator for the spot covariance:

$$\hat{\Sigma}_s^{kl} = \frac{n_k}{k_n}\left(\overline{\Sigma}_{s+k_n/n}^{kl} - \overline{\Sigma}_s^{kl}\right), \tag{11}$$

for $s \in (0, 1-k_n/n]$, while for $1-k_n/n < s \leq 1$, $\hat{\Sigma}_s^{kl} := \hat{\Sigma}_{1-k_n/n}^{kl}$. In Section 4 we provide a detailed description of the asymptotic properties of $\hat{\Sigma}_s$ in the case in which $\overline{\Sigma_t}$ is as in Hayashi and Yoshida (2005). The corresponding implied estimator for $\beta^l$ (see equation (9)) is given as

$$\overline{\beta}^l = \sum_{k=1}^{d} \sum_{m=1}^{n_k} w_{t_{m-1}^k}^k \hat{\Sigma}_{t_{m-1}^k}^{lk} (t_m^k - t_{m-1}^k). \tag{12}$$

With the aim to improve the accuracy of the the pre-estimator $\overline{\Sigma}$ of $\Theta$, we consider a $d \times d$ adjustment process $\Delta_s$ and we define

$$\overline{\beta}_\Delta^l = \sum_{k=1}^{d} \sum_{m=1}^{n_k} w_{t_{m-1}^k}^k \left(\hat{\Sigma}_{t_{m-1}^k}^{lk} (t_m^k - t_{m-1}^k) - \Delta_{t_{m-1}^k}^{lk}\right). \tag{13}$$

It will be useful to rewrite (13) using matrix notation. For this, we first gather all spot covariation adjustments into the following vector

$$\delta = (\delta^{11\prime}, \delta^{12\prime}, \ldots, \delta^{1d\prime}, \ldots, \delta^{d1\prime}, \ldots, \delta^{dd\prime})', \tag{14}$$

*In the remainder of the paper, we will omit the subscript when $t = 1$ and similar for other quantities of interest that we define.

where $\delta^{lk} = (\Delta^{lk}_{t^k_0}, \ldots, \Delta^{lk}_{t^k_{n_k-1}})'$. Furthermore, using the notation $0'_m := (\underbrace{0, \ldots, 0}_{m})$ for $m \in \mathbb{N}$, we let $W$ be a $d \times dn$ matrix whose $i$th row is given by

$$W^i = (0'_{(i-1)n}, w', 0'_{(d-i)n}),$$

in which $w$ is an element of $\mathbb{R}^n$ satisfying that

$$w = \left(w^{1\prime}, \ldots, w^{j\prime}, \ldots, w^{d\prime}\right)',$$

with $w^j = (w^j_{t^j_0}, \ldots, w^j_{t^j_{n_j-1}})'$. Under the preceding notation, equation (13) reads as

$$\overline{\beta} - \overline{\beta}_{\Delta} = W\delta. \tag{15}$$

From Equation (13) it follows that the corresponding estimator of $\Theta$ is:

$$\overline{\Sigma} - \overline{\Delta}, \tag{16}$$

in which $\overline{\Delta}^{kl} := \sum_{m=1}^{n_k} \Delta^{kl}_{t^k_{m-1}}$. Observe that

$$\mathrm{vec}(\overline{\Delta}') = \mathcal{A}\delta, \tag{17}$$

where

$$\mathcal{A}^{(i-1)d+j,l} := \begin{cases} 1, & l = (i-1)n + \sum_{k=1}^{j-1} n_k + 1, \ldots, (i-1)n + \sum_{k=1}^{j} n_k; \\ 0, & \text{otherwise}, \end{cases}$$

for $l = 1, \ldots, dn$ and $i, j = 1, \ldots, d$. Finally, it is natural to restrict the adjustment matrices to be symmetric, namely $\overline{\Delta} = \overline{\Delta}'$. The latter can be achieved by requiring that

$$Q\delta := 0_{(d-1)d/2},$$

in which $Q$ is a $\frac{d(d-1)}{2} \times dn$ dimensional matrix defined for $l = 1, \ldots, dn$ and for $i, j = 1, \ldots, d$ with $i > j$, as

$$Q^{\frac{(i-2)(i-1)}{2}+j,l} = \begin{cases} 1, & l = n(j-1) + \sum_{k=1}^{i-1} n_k + 1, \ldots, n(j-1) + \sum_{k=1}^{i} n_k; \\ -1, & l = n(i-1) + \sum_{k=1}^{j-1} n_k + 1, \ldots, n(i-1) + \sum_{k=1}^{j} n_k; \\ 0, & \text{otherwise}. \end{cases}$$

In Appendix 10.1, we illustrate how the use of $Q$ guarantees symmetry on $\overline{\Delta}$.

## 3 Definition of the BAC estimator

The adjustment vector $\delta$ can be calibrated in various ways. The central idea of this paper is that in many cases there is an accurate estimate available for the stock-ETF beta. We call it the target beta and denote this by $\beta_\bullet$. It may be an estimate provided by financial analysts or an accurate estimator based on the availability of ETF price data.

Formally, we seek to find an optimal adjustment process $\Delta$ that satisfies the condition $\overline{\beta}_\Delta = \beta_\bullet$. The latter, according to (15), implies that

$$\overline{\beta} - \beta_\bullet = W\delta.$$

Among the infinite number of possibilities for $\delta$, we look for the smallest weighted adjustment:

$$\sum_{k=1}^{d}\sum_{l=1}^{d} n_k \sum_{m=1}^{n_k} (\Delta^{kl}_{t^l_{m-1}})^2.$$

Note that the factor $n_k$ is added to adjust for trading frequencies with lower number of observations. Moreover, as discussed above, in order to account for symmetry, it must hold that $Q\delta = 0_{(d-1)d/2}$. Thus, the adjustment $\hat{\delta}$ is chosen in such a way that

$$\hat{\delta} = \operatorname*{argmin}_{\delta} \delta' P \delta, \quad \text{s.t.} \quad \begin{cases} W\delta = \overline{\beta} - \beta_\bullet; \\ Q\delta = 0_{(d-1)d/2}, \end{cases} \tag{18}$$

in which

$$P = \operatorname{diag}\left( \underbrace{n_1 1'_{n_1}, \dots, n_i 1'_{n_i}, \dots, n_d 1'_{n_d}}_{\times d} \right), \tag{19}$$

where we have let $1_m := (\underbrace{1, \dots, 1}_{m})$, $m \in \mathbb{N}$.

Before presenting the solution to (18) (which is derived in Appendix 10.2) we introduce more notation. From now on $\bar{W}$ and $\mathcal{Q}$ will denote two matrices with dimensions $d \times d^2$ and $d^2 \times d^2$, respectively. Moreover, the $k$th row of $\bar{W}$ satisfies that

$$\bar{W}^k = \left( 0'_{(k-1)d}, \frac{1}{n_1}\sum_{m=1}^{n_1} w^1_{t^1_{m-1}}, \dots, \frac{1}{n_d}\sum_{m=1}^{n_d} w^d_{t^d_{m-1}}, 0'_{(d-k)d} \right), \tag{20}$$

while the rows of $\mathcal{Q}$ are defined for $i, j = 1, \dots, d$ as

$$\mathcal{Q}^{(i-1)d+j} = \begin{cases} \left(0'_{(i-1)d+j-1}, 1, 0'_{(d-i+1)d-j}\right) + \left(0'_{(j-1)d+i-1}, -1, 0'_{(d-j+1)d-i}\right) & \text{if } i \neq j; \\ 0'_{d^2} & \text{otherwise.} \end{cases} \tag{21}$$

Using the notation introduced above, we have that the solution to the minimum adjustment optimization problem in (18) fulfils that

$$\mathrm{vec}(\overline{\Delta}) = \mathcal{A}\hat{\delta} = L\left(\overline{\beta} - \beta_{\bullet}\right), \tag{22}$$

where

$$L = \left(I_{d^2} - \frac{1}{2}\mathcal{Q}\right)\bar{W}'\left(I_{d^2}\left(\sum_{k=1}^{d}\frac{\sum_{m=1}^{n_k}(w^k_{t^k_{m-1}})^2}{n_k}\right) - \frac{\bar{W}\mathcal{Q}\bar{W}'}{2}\right)^{-1}. \tag{23}$$

**Definition 1.** *Given a pre-estimator $\overline{\Sigma}$ with corresponding stock-ETF beta $\overline{\beta}$ and associated minimum adjustment projection matrix $L$, the BAC estimator with target beta $\beta_{\bullet}$ is*

$$\overline{\Sigma}^{BAC} = \overline{\Sigma} - \overline{\Delta}^{BAC}, \tag{24}$$

*with*

$$\mathrm{vec}(\overline{\Delta}^{BAC}) = L(\overline{\beta} - \beta_{\bullet}). \tag{25}$$

In the upcoming sections, we describe the asymptotic properties of the proposed estimator within the framework of Hayashi and Yoshida (2005). Their estimator is given by

$$\overline{\Sigma}^{kl}_t = \sum_{i,j}\Delta_{i,k}X^k\Delta_{j,l}X^l\mathbf{1}_{(0,t]}(t^k_i \vee t^l_j)\mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset}, \quad 0 \le t \le 1, \tag{26}$$

where we have let

$$I^k_i = (t^k_{i-1}, t^k_i], \quad i = 1, \ldots, n_k, k = 1, \ldots, d \tag{27}$$

and

$$\Delta_{i,k}X = X_{t^k_i} - X_{t^k_{i-1}}, \quad i = 1, \ldots, n_k, k = 1, \ldots, d.$$

## 4 Properties

In this part, we provide the asymptotic distribution of the stock-ETF beta associated to the HY pre-estimator. We would like to emphasize that Theorem 1 below is a simple consequence of our more general result regarding the asymptotic distribution of estimators for $\int_0^1 H_s\Sigma^{kl}_s ds$ constructed via (26). Note that in our framework, $H$ is allowed to be a stochastic process. Thus, the parameter of interest can be seen as a random (linear) functional of the spot volatility. Note that to our knowledge, none of the results available in the literature (see Jacod and Rosenbaum (2013), Li et al. (2019) and references therein) can be applied to our situation. Therefore, our general result (which is presented in Appendix 11) adds a new way to estimate random functionals of the volatility under the presence of non-synchronized observations. We recall to the reader that a sequence of random vectors $(\xi_n)_{n\geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to converge stably in law towards $\xi$ (in symbols $\xi_n \stackrel{s.d}{\to} \xi$), which is defined on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$, say $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$, if for every continuous and bounded function $f$ and any bounded random variable $\chi$ it holds that

$$\mathbb{E}(f(\xi_n)\chi) \to \tilde{\mathbb{E}}(f(\xi)\chi),$$

where $\tilde{\mathbb{E}}$ denotes expectation w.r.t. $\tilde{\mathbb{P}}$.

**Assumption 1.** *For $k = 1, \ldots, d$, the process $X^k$ is observed at times $t_i^k = \frac{i}{n_k}$, $i = 0, 1, \ldots, n_k$. Moreover, $n_k \to \infty$ and*

$$\frac{n_k}{n} \to 1, \;\; n = \sum_{l=1}^{d} n_l.$$

**Theorem 1.** *Assume that $X$ is given by (1) and let Assumption 1 holds. If $k_n^2/n \to 0$, then as $n \to \infty$*

$$\sqrt{n}\left(\overline{\beta} - \beta\right) \stackrel{s.d}{\to} MN\left(0, \Psi\right),$$

*where*

$$\Psi = \int_0^1 \Sigma_s (w_s' \Sigma_s w_s') ds + \int_0^1 \left[ \left(\Sigma_s w_s\right) \left(\Sigma_s w_s\right)' \right] ds.$$

**Proposition 1.** *Letting $\beta_{\bullet} = \beta$, it follows that, under the assumptions of Theorem 1, as $n \to \infty$*

$$\sqrt{n}\, \mathrm{vec}\left(\overline{\Delta}^{BAC}\right) \stackrel{s.d}{\to} MN\left(0, L_{\infty} \Psi L_{\infty}'\right), \tag{28}$$

*where $L_{\infty}$ is*

$$L_{\infty} = \left(I_{d^2} - \frac{1}{2}\mathcal{Q}\right) W_{\infty}' \left(I_{d^2} \left(\int_0^1 w_s' w_s \, ds\right) - \frac{\bar{W}_{\infty} \mathcal{Q} \bar{W}_{\infty}'}{2}\right)^{-1} \tag{29}$$

*and $\bar{W}_{\infty}$ is given by rows*

$$\{\bar{W}_{\infty}\}_i = \left(0_{(i-1)d}', \int_0^1 w_s^1 \, ds, \ldots, \int_0^1 w_s^d \, ds, 0_{(d-i)d}'\right). \tag{30}$$

**Remark 1.** *Assumption 1 can be replaced by the weaker condition*

$$\frac{n_k}{n} \to \theta_k \in (0, 1].$$

*In this situation, the statement of Theorem 1 still holds when we subtitute $\Psi$ by the matrix*

$$\tilde{\Psi}^{k'l'} = \sum_{k,l} \left\{ \int_0^1 w_s^k w_s^l \gamma^{kk',ll'} \Sigma_s^{kk'} \Sigma_s^{ll'} ds + \int_0^1 w_s^k w_s^l \gamma^{kl',lk'} \Sigma_s^{kl'} \Sigma_s^{k'l} \right\} ds,$$

*with* $\gamma^{kk',ll'}, \gamma^{kl',lk'} > 0$ *depending only on* $(\theta_k)_{k=1,\dots,d}$ *for* $k, l, k', l' = 1, \dots, d$.

We now construct an estimator for $\Psi$. To do this, note that since for $k', l' = 1, \dots, d$ we have

$$\Psi^{k'l'} = \sum_{kl}^{d} \sum_{m,p} \left\{ \int_{I_m^k \cap I_p^l} \Sigma_s^{k'l'} w_s^k \Sigma_s^{kl} w_s^j ds + \int_{I_m^k \cap I_p^l} \Sigma_s^{kk'} w_s^k \Sigma_s^{ll'} w_s^l \right\} ds,$$

a feasible estimator for $\Psi^{kl}$ is given by

$$\hat{\Psi}^{k'l'} = \sum_{kl}^{d} \sum_{m,p} \int_{I_m^k \cap I_p^l} \tilde{\Sigma}_{t_{m-1}^k}^{k'l'} w_{t_{m-1}^k}^k \hat{\Sigma}_{t_{m-1}^k}^{kl} w_{t_{p-1}^l}^l ds + \int_{I_m^k \cap I_p^l} \hat{\Sigma}_{t_{m-1}^k}^{kk'} w_{t_{m-1}^k}^k \hat{\Sigma}_{t_{p-1}^l}^{ll'} w_{t_{p-1}^l}^l ds, \tag{31}$$

where $\hat{\Sigma}_{t_{m-1}^k}^{kl}$ is as in (11) and

$$\tilde{\Sigma}_t^{k'l'} = \left\{ \bar{\Sigma}_{t+k_n/n}^{kl} - \bar{\Sigma}_t^{kl} \right\}, \quad 0 \leq t \leq 1,$$

in which $\bar{\Sigma}^{kl}$ is defined as in (26).

**Proposition 2.** *Assume that* $X$ *is given by (1) and let Assumption 1 holds. If* $k_n \to \infty$ *and* $k_n/n \to 0$, *then as* $n \to \infty$

$$\hat{\Psi}^{k'l'} \xrightarrow{\mathbb{P}} \Psi^{k'l'}.$$

## 5 BAC estimation in case of microstructure noise and jumps

The consistency and asymptotic normality result for the BAC adjustment matrix in (18) assumes that prices are generated by the Brownian semimartingale process in (1)-(3). In practice, real-world prices are also affected by price jumps and microstructure noise. In this setting, the BAC adjustment can be expected to still reduce the mean squared error of the covariance estimate when an accurate target beta is used. We propose here some bias adjustments in the BAC formula in order to take into account the effect of microstructure noise and jumps on the pre-estimator and the component weights. The adjustment depends on the pre-estimator.

We first present the solution when the original Hayashi and Yoshida (2005) pre-estimator is used, which is robust neither to jumps nor to microstructure noise. We then discuss the case in which a noise and jump robust pre-estimator is used.

### 5.1 Adjustments when using the Hayashi and Yoshida (2005) pre-estimator

#### 5.1.1 The case of microstructure noise and no price jumps

Consider first the case in which the observed price may deviate from the efficient price leading to a microstructure noise term (see e.g., Zhang et al., 2005; Hansen and Lunde, 2006). We then denote the observed prices by

$\tilde{X}^k_{t^k} = X^k_{t^k} + \zeta^k_{t^k}$.

When the HY estimator is used as pre-estimator, the BAC estimator needs to be adjusted to correct for the bias in the pre-estimator, as shown in Zhang et al. (2005). To do so, we assume the noise terms to have zero mean and constant variance. We also assume them to be uncorrelated either with each other or with the efficient price.

**Assumption 2.** $\mathbb{E}\left(\zeta^k_{t_k}\right)^2 = \sigma^k_\zeta$ *with* $\mathbb{E}\zeta^k_{t^k} = \mathbb{E}\left[\zeta^k_{t^k}\zeta^l_{t^l}\right] = \mathbb{E}\left[\zeta^k_{t^k}X^l_{t^l}\right] = \mathbb{E}\left[\zeta^k_{t^k}X^k_{t^k}\right] = 0$ *for all* $k \neq l$.

Under Assumption 2, the bias in the HY-implied beta for stock $i$ equals its microstructure noise variance multiplied by two times the number of observations. The bias corrected beta is then:

$$\overline{\beta} - 2\,\mathrm{dg}\left(\bar{w}(n_1\sigma^1_\zeta, \ldots, n_d\sigma^d_\zeta)\right), \tag{32}$$

where $\bar{w}$ is the vector of averages of observed weights for all assets corrected for noise bias, defined as

$$\bar{w} = \left(\frac{1}{n_1}\sum_{i=1}^{n_1}\tilde{w}^1_{t^1_i}\left(1+\frac{\sigma^1_\zeta}{2}\right)^{-1}, \ldots, \frac{1}{n_d}\sum_{i=1}^{n_d}\tilde{w}^d_{t^d_i}\left(1+\frac{\sigma^d_\zeta}{2}\right)^{-1}\right)'.$$

The bias correction is obtained using a second order Taylor series expansion for noise contaminated weights, namely $\tilde{w}^k_t = a^k_t\exp(X^k_t + \zeta^k_t) \approx w^k_t(1+\zeta^k_t + (\zeta^k_t)^2/2)$. The same correction is needed for the weights in the projection matrix:

$$\tilde{L} = \left(I - \frac{\mathcal{Q}}{2}\right)\bar{W}'\left(\left(\sum_{k=1}^{d}\frac{\sum_{i=1}^{n_k}(\tilde{w}^k_{t^k_i})^2}{n_k\left(1+2\sigma^k_\zeta\right)}\right)I - \frac{\bar{W}\mathcal{Q}\bar{W}'}{2}\right)^{-1}, \tag{33}$$

where the rows of $\bar{W}$ are given by

$$\bar{W}_k = \left(0'_{(k-1)d}, \frac{\sum_{i=1}^{n_1}\tilde{w}^1_{t^1_i}}{n_1}\left(1+\frac{\sigma^1_\zeta}{2}\right)^{-1}, \ldots, \frac{\sum_{i=1}^{n_d}\tilde{w}^d_{t^d_i}}{n_d}\left(1+\frac{\sigma^d_\zeta}{2}\right)^{-1}, 0'_{(d-k)d}\right).$$

In the simulation study and in the empirical application, we estimate the noise variance vector $\boldsymbol{\sigma}_\zeta$ as in Zhang et al. (2005):

$$\hat{\sigma}^k_\zeta = \max\left(\frac{\overline{\Sigma}^{kk}_1 - \tilde{\overline{\Sigma}}^{kk}_1}{2n_k}, 0\right), \tag{34}$$

where $\overline{\Sigma}^{kk}$ is the tick-by-tick realized variance for asset $k$ and $\tilde{\overline{\Sigma}}^{kk}$ is the two-time scale estimator of the integrated variance using a combination of $K$ and $J$ step apart returns, with $K = 30$ and $J = 1$.

### 5.1.2 The case of microstructure noise and jumps

When using the Hayashi and Yoshida (2005) approach, the BAC estimator needs also adjustment in case of price jumps. To formalize this, we first generalize the price process in (1) by including jumps as in Hounyo (2017)

$$X_t^k = X_0^k + \int_0^t \mu_s^k ds + \int_0^t \sigma_s^k dB_s + J_t^k, \quad J_t^k = \sum_{s \le t} \Delta X_s^k, \tag{35}$$

where the jump of $X$ at time $s$ is denoted by $\Delta X_s = X_s - X_{s-}$, $X_{s-} = \lim_{t \to s, t<s} X_t$, and other definitions are the same as in (1). As our beta adjustment approach by its nature is multivariate, a multivariate filter is needed. As proposed in Mancini (2009), we detect the price changes affected by jumps by comparing them with a multiple $\kappa$ of an estimate of the local variance,

$$F_k = \bigcup_i (t_{i-1}^k, t_i{}^k] \quad s.t. \quad (\tilde{X}_{t_i^k}^k - \tilde{X}_{t_{i-1}^k}^k)^2 > \kappa s_{t_i^k}^k + 2\sigma_\zeta^k, \tag{36}$$

where $s_{t_i}^k$ is a jump robust estimate of the quadratic variation of of $X_t$ on the interval $(t_{i-1}^k; t_i^k]$. We join intervals where jumps are detected to filter them out accordingly as

$$F = \bigcup_k F_k, \tag{37}$$

removing the returns computed on an interval for which a jump has been detected in any of the $d$ prices[†]. We then filter away the jumps from the HY pre-estimator in (26) as follows:

$$\frac{\overline{\Sigma}^{kl} - \sum_{i,j} (\tilde{X}_{t_i^k}^k - \tilde{X}_{t_{i-1}^k}^k)(\tilde{X}_{t_j^l}^l - \tilde{X}_{t_{j-1}^l}^l) \mathbf{1}_{[t_{j-1}^l, t_j^l] \cap [t_{i-1}^k, t_i^k] \cap F \neq 0}}{1 - \sum_{i,j} ((\min(t_i^k, t_j^l) - \max(t_{i-1}^k, t_{j-1}^l)) \mathbf{1}_{[t_{j-1}^l, t_j^l] \cap [t_{i-1}^k, t_i^k] \cap F \neq 0}}. \tag{38}$$

In the simulation and empirical application we set the threshold parameter $\kappa$ to 25, which is sufficiently high to distinguish between the jumps and Brownian motion driven price fluctuations as in Corsi et al. (2010) and Boudt et al. (2011).

## 5.2 Other pre-estimators

In the presence of jumps, we always recommend filtering out the jumps from the pre-estimator using the jump detection rule in (36)-(38). This multivariate filter aligns the jump detection used for the multivariate covariance estimation and the pairwise beta estimation used as target in the BAC estimator.

In the presence of microstructure noise, the pre-estimator does not necessarily need to be adjusted. The BAC adjustment in (22) may remain useful for non-robust pre-estimators provided that the target beta is accurately

[†]In the simulation study and empirical application, we estimate the local variance as in (11) using MedRV of Andersen et al. (2012) as an auxiliary estimator for $\overline{\Sigma}_t^{kk}$, owing to its computational simplicity and robustness to zero returns and price jumps.

calibrated. The BAC adjustment can then be expected to reduce the bias due to noise. We illustrate this in the simulation setting when the realized covariance matrix is used as pre-estimator.

When the pre-estimator is robust to microstructure noise, we recommend using (22) with projection matrix as in (33) where the average weights are corrected for the noise. We illustrate this in the simulation study for the two-time scale estimator of Zhang (2011).

# 6 Estimation of target beta

The theoretical results in Section 4 assume the oracle situation in which the target beta ($\beta_\bullet$) equals the true beta ($\beta$). The assumption of knowing $\beta$ may seem restrictive. In practice, we can expect to have accurate estimates of $\beta$ for several reasons. The first reason is that $\beta$ has $d$ unknowns while $\Theta$ has $d(d+1)/2$ unknown parameters. The second reason is that the accuracy of an estimate of $\Theta$ requires synchronizing all $d$ series, while to estimate $\beta$ we only need a synchronization of the stock price series with the ETF log-price process denoted by $Y$. In our case of interest, the ETF price series is liquid.

## 6.1 The case of no noise and no jumps

Let $\mathbb{T}^{(Y)} = \left\{0 = t_0^Y < \cdots < t_{n_Y}^Y \leq 1\right\}$ be the observation times for the ETF log-price $Y$. The pairwise Hayashi and Yoshida (2005) stock-ETF beta estimator for asset $l$ over the interval $[0,t]$ is given by:

$$\overline{\beta}_t^{lY} = \sum_{i,j} \Delta_{i,l} X^l \Delta_{j,Y} \exp(Y) \mathbf{1}_{(0,t]}(t_i^l \vee t_j^Y) \mathbf{1}_{I_i^l \cap I_j^Y \neq \emptyset}, \;\; 0 \leq t \leq 1, \tag{39}$$

where $i = 1, \ldots n_l$, $j = 1, \ldots, n_Y$ and $I_i^l$, $I_j^Y$ are as in (27).

We now study how to improve the pairwise estimate of the beta vector when a highly accurate estimate is available for the quadratic variation of the ETF. The improved beta estimate is based on the following result (proof is given in Appendix).

**Proposition 3.** *The quadratic variation of the synthetic ETF log-price $Y^*$ in* (8) *satisfies that*

$$d[Y^*]_s = \sum_{l=1}^{d} \frac{w_s^l}{\exp(2Y_s^*)} d\beta_s^l, \tag{40}$$

*where*

$$\beta_t^l = \sum_{l=1}^{d} \int_0^t w_s^l \Sigma_s^{lk} ds. \tag{41}$$

Based on Proposition 3 we have the following estimate of the quadratic variation of the ETF log-price:

$$\overline{\gamma} = \sum_{l=1}^{d} \sum_{m=1}^{n^l} \frac{w^l_{t^l_{m-1}}}{\exp(2Y_{t^l_{m-1}})} \left( \overline{\beta}^{lY}_{t_m} - \overline{\beta}^{lY}_{t_{m-1}} \right). \tag{42}$$

Suppose we have an alternative highly efficient estimate that we denote by $\gamma_\bullet$ and that the beta estimates need to be adjusted in such a way that the beta-implied variance equals $\gamma_\bullet$. To do so, we propose to determine the adjustment vector $\theta = ((\theta^1_{t^1_1}, \dots, \theta^1_{t^1_{n_1}}), \dots, (\theta^d_{t^d_1}, \dots, \theta^d_{t^d_{n_d}}))'$ of dimension $n$ such that

$$\gamma_\bullet = \sum_{l=1}^{d} \sum_{m=1}^{n^l} \frac{w^l_{t^l_{m-1}}}{\exp(2Y_{t^l_{m-1}})} \left( \overline{\beta}^{lY}_{t_m} - \overline{\beta}^{lY}_{t_m} - \theta^l_{t^l_{m-1}} \right). \tag{43}$$

Similarly as for the beta adjustment of the covariance matrix in (18)-(19), we define the corresponding optimization problem as

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}}\, \theta' D \theta, \tag{44}$$

subject to (43) and where $D = \operatorname{diag}(n_1, \dots, n_d)$.

The resulting variance adjusted beta equals

$$\overline{\beta}^{VAB} = \overline{\beta}^{Y} - \overline{\theta}, \tag{45}$$

where $\overline{\theta}$ is given by

$$\overline{\theta} = \frac{\overline{\gamma} - \gamma_\bullet}{\sum_{j=1}^{d} n_j^{-1} \sum_{m=1}^{n_j} (w^j_{t^j_{m-1}} / \exp(2Y_{t^j_{m-1}}))^2} \left( n_1^{-1} \sum_{m=1}^{n_1} \frac{w^1_{t^1_{m-1}}}{\exp(2Y_{t^1_{m-1}})}, \dots, n_d^{-1} \sum_{m=1}^{n_d} \frac{w^d_{t^d_{m-1}}}{\exp(2Y_{t^d_{m-1}})} \right)'. \tag{46}$$

In the simulation study, we set the target variance to the realized ETF variance when there are no jumps in the ETF and stock prices:

$$\gamma_\bullet = \sum_{i=1}^{n_Y} (Y_{t^Y_i} - Y_{t^Y_{i-1}})^2.$$

## 6.2 The case of microstructure noise and jumps

In the case of the ETF price, we denote by microstructure noise all deviations of the observed log-price from the efficient log-price, namely $Y_{t^Y} = Y^*_{t^Y} + \zeta^Y_{t^Y}$. In contrast to the microstructure noise affecting the covariance-implied beta (denoted by $\overline{\beta}$), the microstructure noise does not induce a bias in the pairwise stock-ETF beta (denoted by $\overline{\beta}^l$) under the assumption that microstructure noise has zero mean and is independent of the efficient stock price and the microstructure noise of the stock prices, *i.e.*, $\mathbb{E}\left[\zeta^Y_{t^Y}\right] = \mathbb{E}\left[\zeta^k_{t^k}\zeta^Y_{t^Y}\right] = \mathbb{E}\left[\zeta^Y_{t^Y} X^k_{t^k}\right] = 0$, for all

$k \neq l$.

Jumps do affect the pairwise stock-ETF beta estimate and need to be removed. To do so, we assume the noise variance to be constant. Let $\sigma_\zeta^Y = E[(\zeta_{t^Y}^Y)^2]$. We detect all ETF jump intervals as in (36):

$$F_Y = \bigcup_i (t_{i-1}^Y, t_i{}^Y] \quad s.t. \quad (Y_{t_i^Y} - Y_{t_{i-1}^Y})^2 > \kappa s_{t_i^Y}^Y + 2\sigma_\zeta^Y.$$

In order to achieve coherence between the jump detection intervals used in the pre-estimator and in the stock-ETF beta estimator, we take the union between the stock jump time intervals $F$ in (37) and the ETF jump time intervals by setting

$$\tilde{F} = F \cup F_Y.$$

The final estimator for the jump corrected Hayashi-Yoshida pre-estimator when the stock-ETF beta is also estimated is then

$$\frac{\overline{\Sigma}^{kl} - \sum_{i,j}(\tilde{X}_{t_i^k}^k - \tilde{X}_{t_{i-1}^k}^k)(\tilde{X}_{t_j^l}^l - \tilde{X}_{t_{j-1}^l}^l)\mathbf{1}_{[t_{j-1}^l, t_j^l] \cap [t_{i-1}^k, t_i^k] \cap \tilde{F} \neq 0}}{1 - \sum_{i,j}((\min(t_i^k, t_j^l) - \max(t_{i-1}^k, t_{j-1}^l))\mathbf{1}_{[t_{j-1}^l, t_j^l] \cap [t_{i-1}^k, t_i^k] \cap \tilde{F} \neq 0}}, \tag{47}$$

and the $l$th element of jump bias adjusted pairwise stock-ETF beta estimate is

$$\frac{\overline{\beta}^{lY} - \sum_{i,j} \Delta_{i,l} \tilde{X}^l \Delta_{j,Y} \exp(Y)\mathbf{1}_{(t_{j-1}^Y, t_j^Y] \cap (t_{i-1}^l, t_i^l] \cap \tilde{F} \neq 0}}{1 - \sum_{i,j}((\min(t_i^l, t_j^Y) - \max(t_{i-1}^l, t_{j-1}^Y))\mathbf{1}_{(t_{j-1}^Y, t_j^Y] \cap (t_{i-1}^l, t_i^l] \cap \tilde{F} \neq 0}} \tag{48}$$

for $l = 1, \ldots, d$. In the simulation study and empirical application, we adjust for the presence of microstructure noise and jumps in the stock and ETF prices by calibrating the target ETF variance as follows:

$$\gamma_\bullet = \sum_{i=1}^{n_Y} \left( (Y_{t_i^Y} - Y_{t_{i-1}^Y})^2 - 2\sigma_\zeta^Y \right) \mathbf{1}_{(t_{i-1}^Y, t_i^Y] \cap \tilde{F} = 0}. \tag{49}$$

## 7 Simulation

We now document the accuracy gains when the Hayashi and Yoshida (2005) realized covariance and the two-time scale estimator proposed by Zhang (2011) are used as pre-estimator. We use here Monte Carlo simulations to investigate the sensitivity of the mean squared error accuracy gains to the properties of the underlying price process, the sampling properties and the target beta calibration method.

### 7.1 Setup

We simulate a $d$-dimensional Brownian semimartingale with jumps using a similar simulation setup as in Barndorff-Nielsen et al. (2011):

$$
\begin{aligned}
\mathrm{d}X_s^k &= \mu^k ds + \mathrm{d}V_s^k + \mathrm{d}F_s + \mathrm{d}J_s^k, \\
\mathrm{d}V_s^k &= \rho_s^k \sigma_s^k \mathrm{d}B_s^k, \\
\mathrm{d}F_s^k &= \sqrt{1-(\rho_s^k)^2}\sigma_s^k \mathrm{d}W_s,
\end{aligned}
$$

where $F^k$ is the common factor, $W_s \perp\!\!\!\perp B_s^k$, $\mu^k$ is the constant drift and $\sigma_s^k$ is a stochastic volatility process

$$\sigma_s^k = \exp(\beta_0^k + \beta_1^k \varrho_s^k),$$

$$\mathrm{d}\varrho_s^k = \alpha^k \varrho_s^k \mathrm{d}s + \mathrm{d}B_s^k,$$

where $B^k$ is a standard Brownian motion and $J^k$ is the jump process in (35). The parameters are set as in Barndorff-Nielsen et al. (2011), namely $(\mu^k, \beta_0^k, \beta_1^k, \alpha^k, \rho^k) = (0.03, -5/16, 1/8, -1/40, -0.3)$. In the case of jumps, we simulate them as independent jumps with arrivals driven by a Poisson process with frequency of on average of two jumps per day per asset. The size of the jumps equals $M = 10$ times the average spot volatility of the day multiplied by a uniform random variable drawn from a uniform distribution on the interval $([-2,-1] \cup [1,2])$. In the absence of noise the ETF price is modeled as a logarithm of the sum of the prices of the components, as in (8). The unit interval $[0,1]$ corresponds to one day of 7.5 hours of shares trading. We simulated prices at the frequency of 100 milliseconds, making up to $N = 7.5 \times 60 \times 60/0.1 = 270000$ intervals per day. We assume that the ETF prices are observed at this frequency. Stock prices are observed at a lower frequency. We generate the observation times for the $d$ stocks using an exponential distribution for the inter-trade durations with rates equal to

$$\lambda_k = \lambda_1 + \exp\left(\nu \frac{k-d}{d-1}\right)(\lambda_d - \lambda_1), \quad \forall k = 1, \ldots, d, \tag{50}$$

where $\nu = 10$, $\lambda_1 = 2700$ and $\lambda_d = \lambda_Y = 270000$.

Alongside the noise-free setup we also run simulations with microstructure, adding a noise term to the log-prices

$$\tilde{X}_{t_i^k}^k = X_{t_i^k}^k + \zeta_{t_i^k}^k,$$

$$Y_{t_i^Y} = \log\left(\sum_{k=1}^{d} \exp(X_{t_i^Y}^k)\right) + \zeta_{t_i^Y}^Y.$$

The microstructure noise terms $\zeta_{t_i^k}^k$ and $\zeta_{t_i^Y}^Y$ are simulated as i.i.d. from a normal distribution with zero mean and variance such that total noise variance equals $\kappa$ times the integrated variance, where $\kappa$ is calibrated at 8.5%, corresponding to the median value found on the empirical data. As a sensitivity analysis, we let kappa vary between 0 and 0.2.

### 7.2 Analysis

We simulate $S = 1000$ days. For each day, we first compute three pre-estimators: the traditional realized covariance estimator (RC), the two-time scale estimator (TSC) of Zhang (2011) and the Hayashi-Yoshida (HY) estimator. Each estimator is constructed using a pairwise approach. RC and TSC use refresh-time sampling. RC is synchronised using refresh-time at the highest frequency available on the interval $(0, 1]$. TSC is implemented as in Zhang (2011). For TSC we set the short and long window sizes for the two time scales as $J = 1$ and $K = 30$, respectively. The HY estimator is computed as in (26).

For each pre-estimator we then compute the corresponding BAC estimate obtained using three possible beta estimates: the oracle beta ($\beta$), the pairwise beta ($\overline{\beta}^Y$) and the variance adjusted beta ($\overline{\beta}^{VAB}$).

In total, we then have nine integrated covariance estimates per day of simulated prices. We compare the accuracy of the estimators using the mean squared error defined as follows:

$$MSE = \frac{1}{S} \sum_{i=1}^{S} \frac{\mathrm{tr}\left(\left(\hat{\Theta}_i - \Theta_i\right)' \left(\hat{\Theta}_i - \Theta_i\right)\right)}{\mathrm{tr}\left(\Theta_i' \Theta_i\right)}, \tag{51}$$

where $S$ is the number of replications and $\Theta_i$ is the true integrated covariance in replication $i$, while $\hat{\Theta}_i$ is its estimate. We express the improvement in MSE achieved by the BAC estimator using the percentage relative improvement in average loss (PRIAL) frequently used in the shrinkage literature:

$$PRIAL_{BAC} = \frac{MSE_{PRE} - MSE_{BAC}}{MSE_{PRE}}, \tag{52}$$

where $MSE_{PRE}$ is the MSE of the pre-estimator.

### 7.3 Results

Table 1 reports the mean squared estimation error for the pre-estimator and the PRIAL values for the BAC estimators. The PRIAL is always computed versus the MSE of the pre-estimator used. As target beta, we take the oracle beta ($\beta_\bullet = \beta$), the pairwise estimated stock-ETF beta ($\beta_\bullet = \overline{\beta}^Y$) and the variance adjusted stock-ETF beta ($\beta_\bullet = \overline{\beta}^{VAB}$). Our contribution is to improve further the performance of the pre-estimator by adjusting them such that the implied stock-ETF beta correspond to a target beta. We report the results for three scenarios: (i) no noise and no jumps (panel A), (ii) noise and no jumps (panel B), and (iii) noise and jumps (panel C). For the scenarios with noise and/or jumps we perform the same bias corrections for the pre-estimator, as explained in Sections 5.1.1 and 5.1.2. For each panel, we consider three dimensions ($d = 10$, $d = 30$ and $d = 100$) and three estimators (RC, TSC and HY).

Note first that the performance pattern of BAC with respect to choice of pre-estimators and betas doesn't vary much across Panels A, B and C. We will therefore discuss only panel C which covers the realistic case of prices being affected by both microstructure noise and price jumps.

As pre-estimator, the HY estimator stands out in terms of lowest MSE (0.027 for $d = 10$ and 0.030 for $d = 100$). The RC and TSC estimators implemented with refresh time sampling have a larger MSE. Their MSE is around 200 (respectively 10) times the MSE of the HY estimator.

All PRIAL values are positive indicating the gains in accuracy of the BAC estimator compared to the pre-estimator. The accuracy of the target beta clearly matters. The PRIAL values are between 56% and 72% when using the oracle beta ($\beta_\bullet = \beta$), between 10% and 74% for the pairwise estimated stock-ETF beta ($\beta_\bullet = \overline{\beta}^Y$), and between 16% and 74% when targeting the variance adjusted stock-ETF beta ($\beta_\bullet = \overline{\beta}^{VAB}$). The improvement becomes slightly smaller when the dimension $d$ increases, which is expected as there are more elements to adjust per element of the beta differential vector.

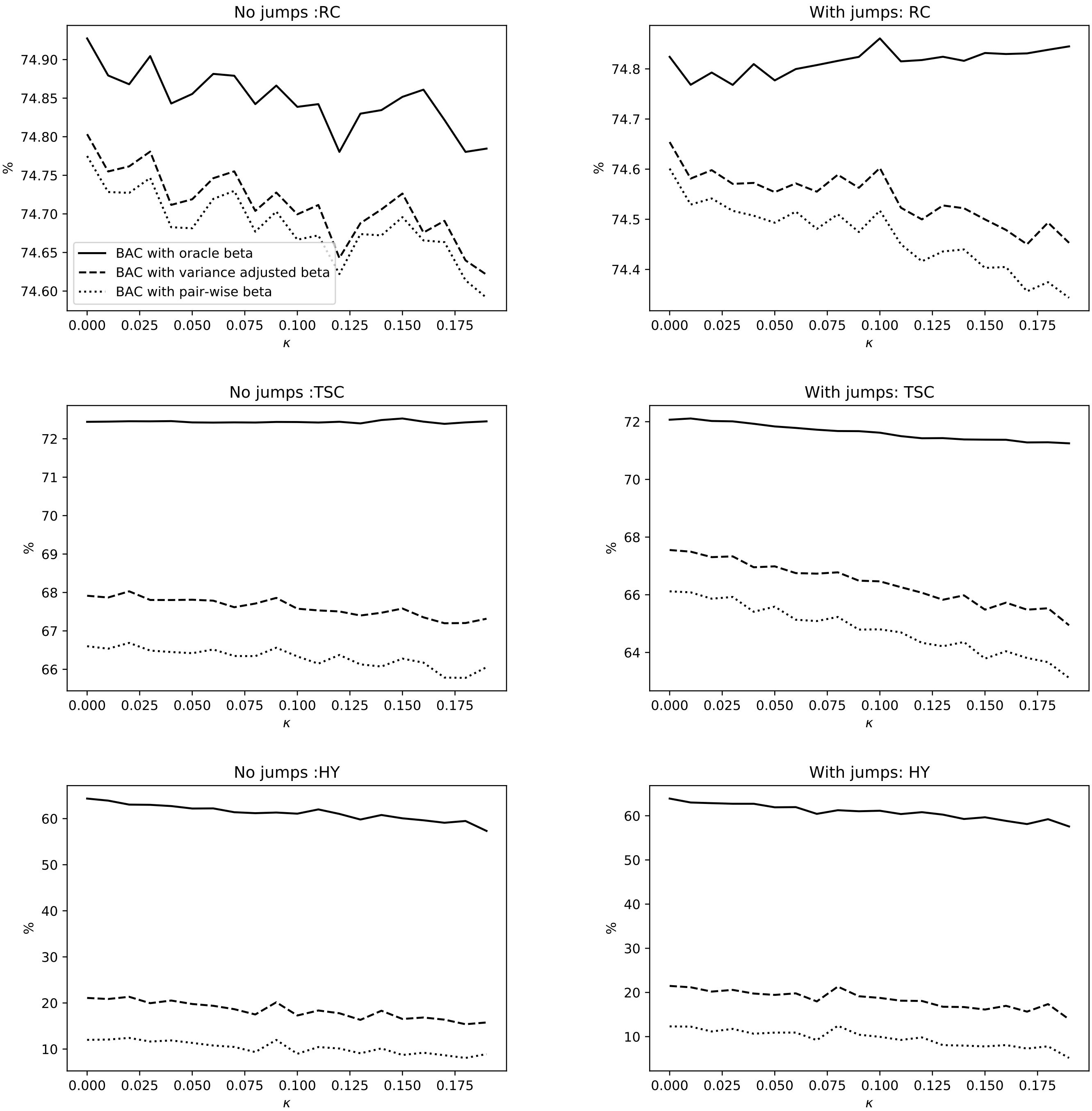
So far we have discussed the results in Table 1 where we have either absence of microstructure noise or when the microstructure noise to asset noise-free variance ratio $\kappa$ is equal to 0.085. In Figure 1 we analyze this at the more granular level by letting the noise variance parameter $\kappa$ vary from 0 to 0.2. We can see that, while higher levels of microstructure noise variance reduce the size of the MSE improvement, the overall conclusion remains that the BAC estimator leads to economically significant improvements of accuracy with PRIAL values that are above 57%, 5% and 14% when using the oracle beta ($\beta_\bullet = \beta$), the pairwise estimated stock-ETF beta ($\beta_\bullet = \overline{\beta}^Y$) and the variance adjusted stock-ETF beta ($\beta_\bullet = \overline{\beta}^{VAB}$), respectively.

Table 1: MSE and PRIAL of integrated covariance estimates

| | Pre-estimator | $MSE_{PRE}$ (in %) | PRIAL (in %) for BAC with $\beta_\bullet =$ $\beta$ | $\overline{\beta}^Y$ | $\overline{\beta}^{VAB}$ |
|---|---|---|---|---|---|
| Panel A: No noise and no jumps | | | | | |
| $d = 10$ | HY | 0.023 | 65.484 | 6.288 | 19.433 |
| | RC | 4.165 | 72.124 | 71.743 | 71.830 |
| | TSC | 0.238 | 74.791 | 68.339 | 70.119 |
| $d = 30$ | HY | 0.026 | 63.260 | 10.478 | 21.418 |
| | RC | 5.437 | 76.418 | 76.161 | 76.210 |
| | TSC | 0.265 | 74.309 | 68.581 | 69.936 |
| $d = 100$ | HY | 0.025 | 57.489 | 9.635 | 18.356 |
| | RC | 5.954 | 74.072 | 73.886 | 73.926 |
| | TSC | 0.277 | 71.216 | 66.256 | 67.398 |
| Panel B: Noise and no jumps ($\kappa = 0.085$) | | | | | |
| $d = 10$ | HY | 0.028 | 64.437 | 13.025 | 21.932 |
| | RC | 4.177 | 72.353 | 72.101 | 72.133 |
| | TSC | 0.236 | 74.700 | 67.668 | 68.928 |
| $d = 30$ | HY | 0.029 | 59.936 | 10.282 | 17.608 |
| | RC | 5.394 | 76.040 | 75.784 | 75.825 |
| | TSC | 0.270 | 74.115 | 68.193 | 69.220 |
| $d = 100$ | HY | 0.028 | 55.817 | 8.752 | 14.959 |
| | RC | 5.950 | 73.750 | 73.548 | 73.574 |
| | TSC | 0.253 | 70.190 | 64.497 | 65.444 |
| Panel C: Noise and jumps ($\kappa = 0.085$) | | | | | |
| $d = 10$ | HY | 0.027 | 62.837 | 13.270 | 21.659 |
| | RC | 4.177 | 71.990 | 71.679 | 71.736 |
| | TSC | 0.240 | 75.804 | 69.480 | 70.730 |
| $d = 30$ | HY | 0.028 | 60.323 | 10.721 | 17.886 |
| | RC | 5.368 | 76.421 | 76.063 | 76.123 |
| | TSC | 0.279 | 73.956 | 68.220 | 69.246 |
| $d = 100$ | HY | 0.030 | 56.314 | 9.766 | 15.805 |
| | RC | 6.089 | 74.451 | 73.879 | 73.965 |
| | TSC | 0.296 | 71.107 | 64.814 | 65.906 |

Note: In Panel A, the HY, RC and TSC pre-estimator are the standard Hayashi-Yoshida, Realized Covariance and Two-time Scale Covariance estimators. In panels B and C, we remove from the HY and RC estimator the bias due to noise, as explained in Subsection 5.1.1. In Panel C, we filter out the returns that are affected by jumps, as explained in Subsection 5.1.2.

Figure 1: Sensitivity of the PRIAL of the BAC estimators to noise variance




Note: The figures show the effect of noise variance magnitude on the PRIAL of the BAC estimator in the cases of no jumps (left panel) and jumps (right panel). The pre-estimators have been bias-adjusted for the effect of microstructure noise, as explained in Subsection 5.1. In the case of price jumps, these are also filtered out using the jump test explained in Subsection 5.1.2.

# 8 Empirical application

Our paper is motivated by the opportunity to improve realized covariance estimation by exploiting the increasing number of transactions involving exchange traded funds. In this section, we document the BAC adjustment for realized covariance estimation of the stocks for which the market capitalization weighted value is tracked by the Financial Select Sector SPDR Fund, with ticker XLF. The Financial Select Sector SPDR Fund (XLF) is among the most frequently traded ETFs (nasdaq.com, 2019).

We first describe the data and compare the properties of the ETF data and the stock price data. Second, we quantify the magnitude of the BAC adjustment and document its heterogeneity across time and stocks. Third, we show that the adjustment improves the performance of an index tracking investor aiming at tracking the XLF index with a small number of stocks.

## 8.1 Data

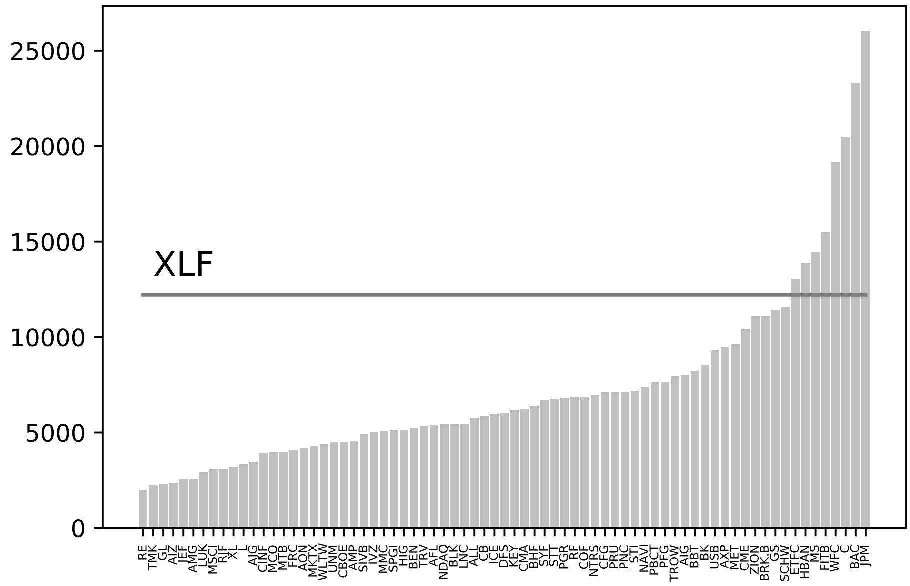
We use two years - from Jan 2 2018 to Dec 31, 2019 - of transaction prices from the Trades and Quotes (TAQ) Millisecond database for the XLF fund transaction prices and its 67-69 components. The amount of investment in the various assets is taken from the CRSP Mutual Funds constituents database. Data cleaning is performed according to recommendations in Barndorff-Nielsen et al. (2009).[‡] We find that, for our sample, the XLF ETF tracks the value of a market capitalization weighted portfolio invested in financial sector stocks included in the S&P 500 with a tight tracking error.[§] We refer to citetpetajisto2017inefficiencies for more dicusion on the mechanism of shares redemption and the activity of arbitrageurs ensuring such low tracking errors.

Figure 2 reports the daily average number of cleaned trades for all stocks included in the XLF. It varies between 1987 and 26038 observations per day with a an average (resp. median) value of 7330 (resp. 6091) trades per day. The XLF fund itself has an average frequency of 12211 trades per day. Only eight stocks have a higher number of observations, namely JPMorgan Chase (JPM), Bank of America (BAC), Citigroup (C), Wells Fargo (WFC), Fifth Third Bancorp (FITB), Morgan Stanley (MS), Huntington Bancshares (HBAN) and E*TRADE Financial Corporation (ETFC).

[‡] One exception is that for each stock we take all trades on the two most liquid exchanges instead of only one exchange. This modification substantially increases the number of observations with only little effect on the microstructure noise variance.

[§] On our sample, the relative mispricing between the ETF price ($\exp(Y_t)$) and the weighted average of the most component stock prices obtained using last tick interpolation for every minute. We find that the relative mispricing is economically small. It ranges between -0.19% and 0.34%, with zero mean and median and standard deviation of 0.01% .

Figure 2: Average number of daily cleaned trades for the XLF stocks from Jan 1, 2018-Dec 31, 2019




Note: We show here the average number of trades for the components of the XLF fund. The horizontal line indicates the average number of trades for the XLF fund.

## 8.2 Magnitude of BAC adjustment

The size of the BAC adjustment in (25) is driven by the difference between the pre-estimator implied stock-ETF beta $\overline{\beta}$ and the target beta $\overline{\beta}^Y$ in (39).

We gauge the across-asset variation in Figure 3 where we report for each stock in the XLF funcd the magnitude of the estimated beta-differential for the HY estimator. More specifically, for each stock $k$, we report the following normalized root mean squared adjustment in beta:

$$D^k = c^k \sqrt{\frac{1}{T}\sum_{t=1}^{T}(\overline{\beta}_t^{kY} - \beta_t^k)^2},$$

where $c_k$ is a normalizing constant equal to the inverse average absolute value of beta over the entire period. Results are presented for all components of the XLF fund sorted by frequency of trade, from lowest to highest. The aggregated beta difference is clearly higher on the left side, where less frequently traded instruments are located, implying larger estimation error.

In Figure 4 we show the time series variation in the total magnitude of the BAC adjustment. For each day, we report the norm of the BAC adjustment matrix divided by the norm of the pre-estimator $(\left\|\overline{\Delta}^{BAC}\right\| / \|\Sigma\|)$. We see that there the fluctuations in the magnitude of the adjustment are sizable and that they are serially

Figure 3: Across-asset variation in magnitude of the HY pre-estimator implied stock-ETF beta and the target beta for all XLF stocks sorted from lowest to highest number of average observations per day

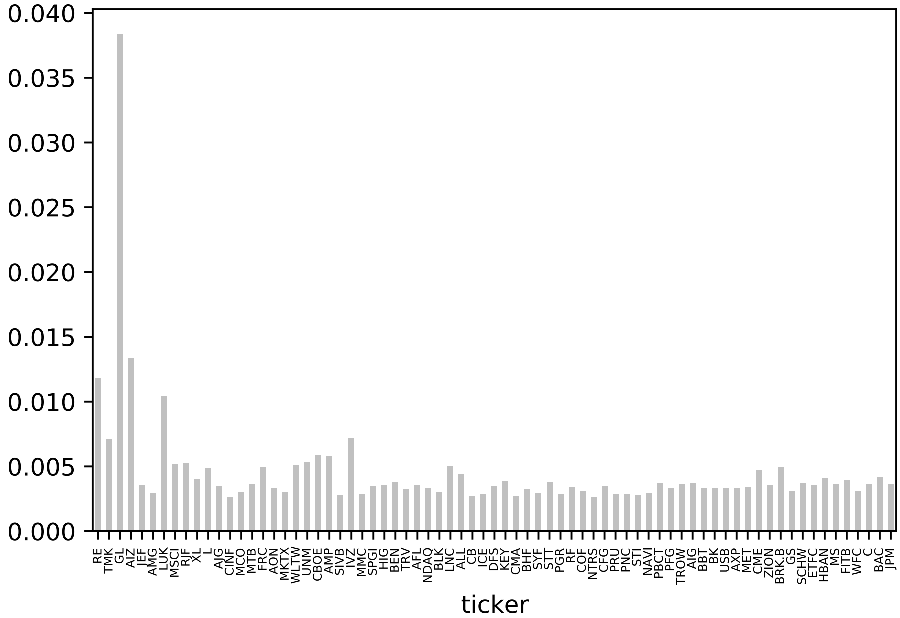
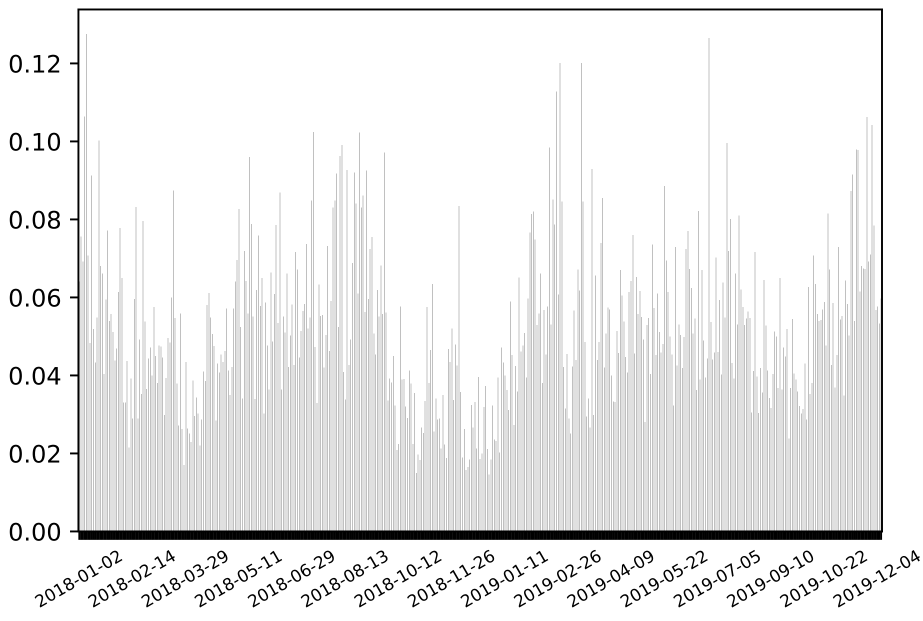



Figure 4: Time series variation in the norm of the BAC adjustment matrix



correlated, indicating that the gains of the BAC adjustment are also time-varying.

### 8.3 Index tracking portfolio

Now we want to evaluate BAC performance on market data via its application to index tracking. Fastrich et al. (2014) describe index tracking as a passive financial strategy that aims at replicating the performance of a given index. They note that full replication using all constituents of the index is often not possible since having many active positions in the tracking portfolio may lead to small and illiquid positions, causing high administrative and transaction costs. The goal of index tracking is to build a portfolio composed of the minority of the components of the index such that it follows the price dynamics of the index as precisly as possible, minimizing the variance of their difference. We show here how realized covariance matrices can be used to construct daily index tracking portfolios by minimizing the covariance-based tracking error. We show that when the performance is evaluated using the next day's realized tracking error, the BAC adjustment improves the performance on average 85 per cent of the days.

#### 8.3.1 Methodology

We consider an investor who aims to track the ETF price index using a subset of $K < d$ stocks included in the ETF portfolio. Let $\mathcal{C}$ be the corresponding feasible set. The investor thus seeks the portfolio of weights $\alpha \in \mathcal{C}$ such that it minimizes the following integrated tracking error variance:

$$TE(\alpha; \Omega) = (\mathbf{1} - \alpha)'\Omega(\mathbf{1} - \alpha),$$

where $\Omega$ is the integrated covariance matrix of the underlying efficient ETF logprice $Y$ and the $K \leq d$ stock prices used to track $Y$:

$$\Omega = \begin{pmatrix} \omega_Y & \omega_{YK} \\ \omega_{YK} & \Theta_K \end{pmatrix}.$$

The $K \times K$ submatrix $\hat{\Theta}_K$ is the integrated covariance matrix of the $K$ log-prices used to track $Y$, $\omega^Y$ is the integrated variance of Y and $\omega^{YK}$ is the $K$-dimensional integrated covariance vector of $Y$ and the $K$ stocks' log-prices. From the first order conditions, we obtain that the minimum tracking error portfolio weights are given by

$$\alpha(\Omega) = \Theta_K^{-1}\omega_{YK}. \tag{53}$$

We now plug in the Hayashi and Yoshida (2005) pre-estimator for the integrated covariance matrix of the $K$ stock prices for each day $t$. Denote these estimates by $\hat{\Theta}_{K,t}$. For $\omega_Y$ and $\omega_{YK}$ we use only the HY estimator and denote the corresponding estimates by $\hat{\omega}_{Y,t}$ and $\hat{\omega}_{YK,t}$. The resulting integrated covariance matrix estimate is:

$$\hat{\Omega}_t = \begin{pmatrix} \hat{\omega}_{Y,t} & \hat{\omega}_{YK,t} \\ \hat{\omega}_{YK,t} & \hat{\Theta}_{K,t} \end{pmatrix}.$$

The corresponding estimated minimum tracking error portfolio is $\alpha(\hat{\Omega}_t)$.[¶] We do the same for the BAC adjusted pre-estimator leading to $\alpha(\hat{\Omega}_t^{BAC})$.

In order to evaluate the tracking error performance of portfolio $\alpha(\hat{\Omega}_t)$ we use the next day's covariance estimate $\hat{\Omega}_{t+1}$. The next days's tracking error is:

$$TE_{t+1}(\alpha(\hat{\Omega}_t)) = \hat{\omega}_{Y,t+1} - 2\hat{\omega}_{YK,t}\hat{\Theta}_{K,t}^{-1}\hat{\omega}_{YK,t+1} + \hat{\omega}_{YK,t}\hat{\Theta}_{K,t}^{-1}\hat{\Theta}_{K,t+1}\hat{\Theta}_{K,t}^{-1}\hat{\omega}_{YK,t}.$$

If $\hat{\Theta}_{K,t} = \hat{\Theta}_{K,t+1}$, then $\alpha(\hat{\Omega}_i)$ delivers the optimal portfolio by construction. We further create for each day $N = 10000$ random sets of $K$ stocks used in the index tracking. We sample $K$ from a uniform distribution between 10 and 30. For each day $t$, we then compute the percentage of subsets for which the BAC adjustment has improved the tracking error:

$$G_t = \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}_{\left(TE_{t+1}(\alpha(\hat{\Omega}_{t,i})) - TE_{t+1}(\alpha(\hat{\Omega}_{t,i}^{BAC}))\right)>0}, \tag{54}$$

where the portfolio weights are computed using estimators $\hat{\Omega}_{t,i}^{HY}$ and $\hat{\Omega}_{t,i}^{BAC}$ of the previous day and the performance is evaluated using the tracking error computed using $\hat{\Omega}_{t+1,i}$ of the day $t+1$. The latter is computed using the 1-minute realized covariance as well as the Hayashi-Yoshida covariance estimator using all trades.

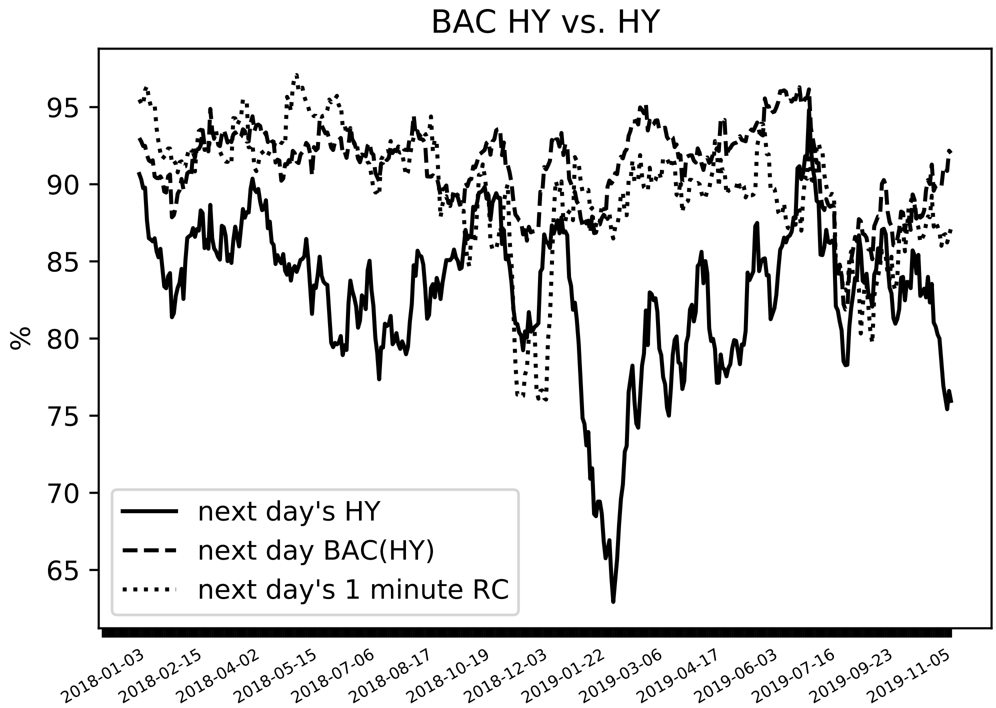
### 8.3.2 Results

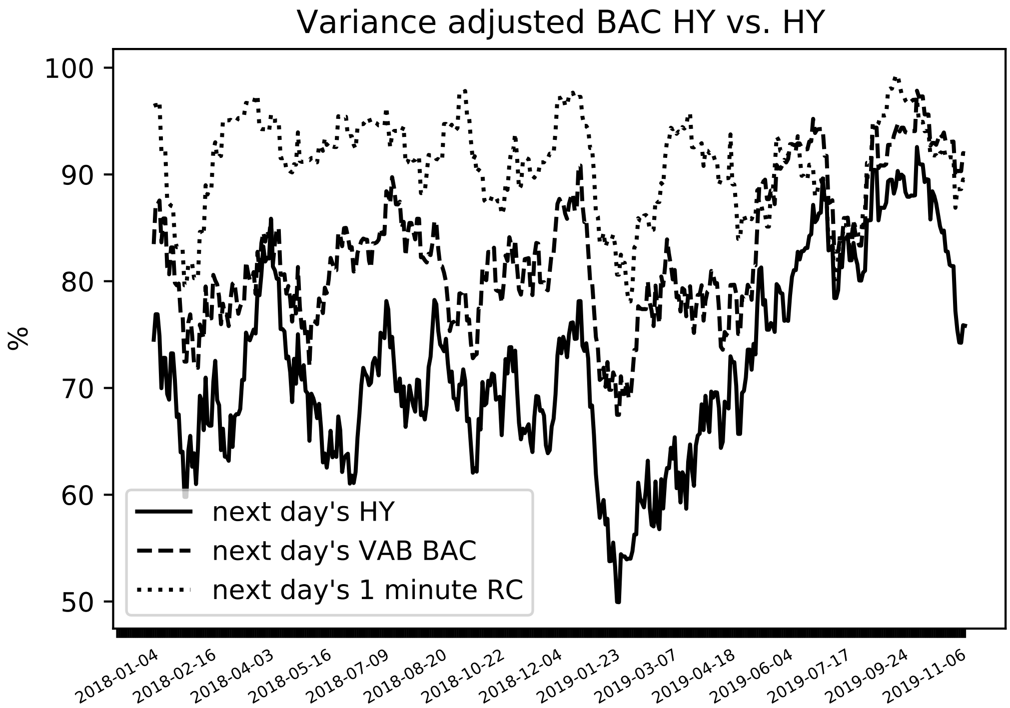
We now use the next day's realized covariance to evaluate the gains obtained using the BAC estimator in terms of achieving a low tracking error portfolio. The evaluation period ranges from Jan 1, 2018 to Dec 31, 2019. Excluding dates with missing data and dates of re-balancing, we have in total 442 days. The results are presented in Figure 5, where we plot the 10-day moving averages for both pairs of estimators, comparing BAC $HY$ against $HY$ and variance adjusted BAC $HY$ against $HY$.

In the top plot, we can see that the BAC estimator outperforms the pre-estimator every day for over 84% of the random subsets considered when we use the next day's HY estimator to evaluate the trackingerror. In the bottom plot, we can see that if we use the variance adjusted beta as target beta, the outperformance of the BAC estimator remains but is less outspoken. However, if we gauge the performance based on the next day's 1-minute realized covariance, then the BAC estimator with variance adjusted beta as target beta outperforms perfoms simiilarly as the BAC estimator with the pairwise estimate as target beta.

[¶]When the pre-estimator or its BAC adjustment is not positive definite, we perform a spectral decomposition based regularization as in Aït-Sahalia et al. (2010) and Fan et al. (2012).

Figure 5: Percentage of outcomes where the BAC estimator-based tracking portfolio has a lower next-day's tracking error (evaluated using next day's HY, BAC or 1-min RC) than the tracking portfolio based on the HY pre-estimator




Note: The figure shows the 10-day moving average of the percentage of outcomes where the minimum tracking error portfolio obtained using the BAC estimator outperforms its counterpart obtained using the HY pre-estimator. Portfolios are formed using the previous day covariance matrix estimate and performance is evaluated using the HY, BAC and fixed grid one-minute RC for the next day.

While it remains a topic for further research to obtain even better estimates for the stock-ETF beta (and possibly exploit expert opinion), we can conclude from the empirical analysis that the BAC adjustment with the pairwise stock-ETF beta yields improved minimum variance tracking error portfolio in the vast majority of the random subsets considered.

# 9 Conclusion

Over the past decade, the trading frequency of several Exchange Traded Funds (ETFs) has surpassed the frequency at which many of their component stocks trade. In this paper, we show that this trend has a positive spillover effect in terms of improved covariance estimation of the underlying stock returns. We develop an econometric framework to exploit the information value in the highfrequency comovement between stock and ETF prices for the estimation of the covariation between stock prices over a fixed time interval.

The proposed Beta Adjusted Covariance estimator improves a pre-estimator in such a way that the implied stock-ETF beta equals a target value. The latter can either be based on pairwise estimation using stock and ETF prices or be defined using expert opinion. We develop the asymptotic theory for the stock-ETF beta associated to the Hayashi and Yoshida (2005) pre-estimator. In the simulation study, we show that the accuracy gains are over 50% in the case in which the target value for the stock-ETF beta is set by an expert to the oracle beta that is assumed to be free from estimation error. The accuracy gains remain economically significant when the target beta is estimated using ETF prices and stock prices. The empirical application on Trades and Quotes millisecond transaction data demonstrates the usefulness of the BAC adjustment for an investor aiming at tracking an investment index with a small number of stocks.

To help practitioners and academics to implement our methodology in practice, we have included the open source implementation of the BAC estimator in the R package highfrequency (Boudt et al., 2021) and the Python package bacpack (Dragun et al., 2021).

# 10 Appendix 1: Derivation of the BAC estimator

## 10.1 Example of $Q$ matrix

The $nd$-dimensional vector $\delta$ corresponds to the adjustment to the $n$ spot covariances estimated using the pre-estimator. We use the $d(d-1)/2 \times dn$ matrix $Q$ to make sure that symmetry in the adjusted covariance is guaranteed by imposing $Q\delta = 0_{d(d-1)/2}$. To illustrate this, suppose that $d = 2$. In this situation, we require that $\bar{\Delta}^{12} = \bar{\Delta}^{21}$ which, is equivalent to having that

$$\sum_{l=1}^{n_2} \delta_l^{12} = \sum_{l=1}^{n_1} \delta_l^{21}.$$

Since $Q = \begin{bmatrix} 0'_{n_1} & 1'_{n_2} & -1'_{n_1} & 0'_{n_2} \end{bmatrix}$, it follows that

$$Q\delta = \sum_{l=1}^{n_2} \delta_l^{12} - \sum_{l=1}^{n_1} \delta_l^{21}.$$

We conclude easily from this that $\bar{\Delta}^{12} = \bar{\Delta}^{21}$ if and only if $Q\delta = 0$.

## 10.2 Proof of Equation (22)

Let us start by considering the Lagrangian corresponding to the optimization problem (18). Plainly,

$$\mathcal{L} = \delta' P \delta - \left[ (W', Q')'\delta - \left( (\overline{\beta} - \beta_\bullet)', 0'_{(d-1)d/2} \right)' \right]' \lambda,$$

with $\lambda$ as the vector of Lagrange multipliers. Thus, the $n = \sum_{k=1}^{d} n_k$ first order conditions for the elements of $\delta$ are:

$$\frac{\partial \mathcal{L}}{\partial \delta} = 2P\delta - (W', Q')\lambda = 0_n. \tag{55}$$

For the $d \times 1$ Lagrangian multipliers $\lambda_i$, we have:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = (W', Q')'\delta - \left( (\overline{\beta} - \beta_\bullet)', 0'_{(d-1)d/2} \right)' = 0_d. \tag{56}$$

From (55) and (56) we obtain:

$$(W', Q')' P^{-1} (W', Q')\lambda = 2 \ \left( (\overline{\beta} - \beta_\bullet)', 0'_{(d-1)d/2} \right)'.$$

Applying the previous relation to (56) and using standard formulas for the inverse of block matrices, we get that $\delta$ equals

$$P^{-1}\begin{pmatrix} D & -DWP^{-1}Q'\left(QP^{-1}Q'\right)^{-1} \\ -\left(QP^{-1}Q'\right)^{-1}QP^{-1}W'D & \left(QP^{-1}Q'\right)^{-1}+\left(QP^{-1}Q'\right)^{-1}QP^{-1}W'DWP^{-1}Q'\left(QP^{-1}Q'\right)^{-1}\end{pmatrix}$$
$$\times\begin{pmatrix}\overline{\beta}-\beta_{\bullet} \\ 0_{(d-1)d/2}\end{pmatrix}$$

where

$$D=\left(WP^{-1}W'-WP^{-1}Q'\left(QP^{-1}Q'\right)^{-1}QP^{-1}W'\right)^{-1}.$$

Therefore,

$$\begin{aligned}\mathrm{vec}(\overline{\Delta})= & \quad \mathcal{A}P^{-1}\begin{pmatrix}W\\Q\end{pmatrix}'\begin{pmatrix}D\\-\left(QP^{-1}Q'\right)^{-1}QP^{-1}W'D\end{pmatrix}P^{-1}\begin{pmatrix}W\\Q\end{pmatrix}'\begin{pmatrix}D\\-\left(QP^{-1}Q'\right)^{-1}QP^{-1}W'D\end{pmatrix}\left(\overline{\beta}-\beta_{\bullet}\right)\\ = & \quad \mathcal{A}P^{-1}\left(I-Q'\left(QP^{-1}Q'\right)^{-1}QP^{-1}\right)W'D\left(\overline{\beta}-\beta_{\bullet}\right).\end{aligned}$$

Thus, it only remains to show that

$$L=\mathcal{A}P^{-1}\left(I-Q'\left(QP^{-1}Q'\right)^{-1}QP^{-1}\right)W'\left(WP^{-1}W'-WP^{-1}Q'\left(QP^{-1}Q'\right)^{-1}QP^{-1}W'\right)^{-1}, \tag{57}$$

with $L$ as in (23). To do this, observe first that

$$QP^{-1}Q'=2I_{d^2};\qquad \mathcal{A}P^{-1}W'=\bar{W}'. \tag{58}$$

Indeed, by using the definition of $Q$, it is easy to see that

$$\begin{aligned}\left(QQ'\right)^{\frac{(i-2)(i-1)}{2}+j,\frac{(i'-2)(i'-1)}{2}+j'} =&\frac{1}{n_i}\sum_{l=(j-1)n+\sum_{k=1}^{i-1}n_k+1}^{(j-1)n+\sum_{k=1}^{i}n_k}Q^{\frac{(i'-2)(i'-1)}{2}+j',l}\\ &-\frac{1}{n_j}\sum_{l=(i-1)n+\sum_{k=1}^{j-1}n_k+1}^{(i-1)n+\sum_{k=1}^{j}n_k}Q^{\frac{(i'-2)(i'-1)}{2}+j',l}\\ =&(\alpha^{ii'}\alpha^{jj'}-\alpha^{ij'}\alpha^{ji'})-(\alpha^{ij'}\alpha^{ji'}-\alpha^{ii'}\alpha^{jj'})\end{aligned}$$

for all $i,j,i',j'=1,\ldots,d$, $i>j$, $i'>j'$. Note that if $i=j'$ and $j=i'$, we would have that $j>i$, which is absurd. Therefore,

$$\left(QQ'\right)^{\frac{(i-2)(i-1)}{2}+j,\frac{(i'-2)(i'-1)}{2}+j'}=2\alpha^{ii'}\alpha^{jj'}=2I_{d^2}^{\frac{(i-2)(i-1)}{2}+j,\frac{(i'-2)(i'-1)}{2}+j'}.$$

Similar arguments can be used to deduce that for all $m, r, j = 1, \ldots, d$

$$\begin{aligned}\sum_{l=1}^{nd}(1/P^{ll})\mathcal{A}^{(m-1)d+r,l}W^{k,l} &= \frac{1}{n_r}\sum_{l=(m-1)n+\sum_{x=1}^{r-1}n_x+1}^{(m-1)n+\sum_{x=1}^{r}n_x} W^{k,l} \\ &= \frac{\alpha^{mk}}{n_r}\sum_{m=1}^{n_r} w^r_{t^r_{m-1}} = \bar{W}^{k,(m-1)d+r},\end{aligned}$$

which shows the validity of (58). Applying the latter to the right-hand side of (57) allows us to conclude that (22) holds if and only if

$$L = \left(\bar{W}' - \frac{1}{2}\mathcal{A}P^{-1}Q'QP^{-1}W'\right)\left(WP^{-1}W' - \frac{1}{2}WP^{-1}Q'QP^{-1}W'\right)^{-1}. \tag{59}$$

Trivially, $WP^{-1}W' = I_{d^2}\left(\sum_{y=1}^{d}\frac{1}{n_y}\sum_{l=1}^{n_y}(w^y_{t^y_{l-1}})^2\right)$. Moreover, in view that for all $i, j, i', j', k, m, r = 1, \ldots, d$ with $i > j$ and $i' > j'$, it holds that

$$\begin{aligned}\sum_{l=1}^{nd}(1/P^{ll})\mathcal{A}^{(m-1)d+r,l}Q^{\frac{(i-2)(i-1)}{2}+j,l} &= \frac{1}{n_r}\sum_{l=(m-1)n+\sum_{x=1}^{r-1}n_x+1}^{(m-1)n+\sum_{x=1}^{r}n_x} Q^{\frac{(i-2)(i-1)}{2}+j,l} \\ &= \alpha^{mj}\alpha^{ri} - \alpha^{mi}\alpha^{rj},\end{aligned} \tag{60}$$

and

$$\begin{aligned}\sum_{l=1}^{nd}(1/P^{ll})Q^{\frac{(i-2)(i-1)}{2}+j,l}W^{k,l} &= \frac{1}{n_i}\sum_{l=(j-1)n+\sum_{x=1}^{i-1}n_x+1}^{(j-1)n+\sum_{x=1}^{i}n_x} W^{k,l} - \frac{1}{n_j}\sum_{l=(i-1)n+\sum_{x=1}^{j-1}n_x+1}^{(i-1)n+\sum_{x=1}^{j}n_x} W^{k,l} \\ &= \alpha^{kj}\frac{1}{n_i}\sum_{l=1}^{n_i} w^i_{t^i_{l-1}} - \alpha^{ki}\frac{1}{n_j}\sum_{l=1}^{n_j} w^j_{t^j_{l-1}},\end{aligned}$$

in which we have let $\alpha^{kl}$ denote the Dirac's delta measure. We obtain that

$$\begin{aligned}\left(\bar{W}\mathcal{A}P^{-1}Q'\right)^{k,\frac{(i-2)(i-1)}{2}+j} &= \sum_{m=1}^{d}\left(\alpha^{mj}\alpha^{mk}\frac{1}{n_i}\sum_{l=1}^{n_i} w^i_{t^i_{l-1}}\right) - \sum_{m=1}^{d}\left(\alpha^{mj}\alpha^{mi}\frac{1}{n_j}\sum_{l=1}^{n_j} w^j_{t^j_{l-1}}\right) \\ &= \left(QP^{-1}W'\right)^{\frac{(i-2)(i-1)}{2}+j,k}.\end{aligned}$$

Consequently, we can rewrite the right-hand side of (59) as

$$\left(I_{d^2} - \frac{1}{2}\left(\mathcal{A}P^{-1}Q'\right)\left(\mathcal{A}P^{-1}Q'\right)'\right)\bar{W}'\left(I_{d^2}\left(\sum_{y=1}^{d}\frac{1}{n_y}\sum_{l=1}^{n_y}(w^y_{t^y_{l-1}})^2\right) - \frac{1}{2}\bar{W}\left(\mathcal{A}P^{-1}Q'\right)\left(\mathcal{A}P^{-1}Q'\right)'\bar{W}'\right)^{-1}.$$

Therefore, in order to finish the proof, we only need to check that $\left(\mathcal{A}P^{-1}Q'\right)\left(\mathcal{A}P^{-1}Q'\right)' = \mathcal{Q}$, where $\mathcal{Q}$ is as in (21). From (60) we obtain that for all $m, r, m', r' = 1, \ldots, d$

$$\left[\left(\mathcal{A}P^{-1}Q'\right)\left(\mathcal{A}P^{-1}Q'\right)'\right]^{(m-1)d+r,(m'-1)d+r'} = \sum_{j=1}^{d-1}\sum_{i=j+1}^{d}(\alpha^{mj}\alpha^{ri} - \alpha^{mi}\alpha^{rj})(\alpha^{m'j}\alpha^{r'i} - \alpha^{m'i}\alpha^{r'j}),$$

which obviously vanishes when $m = r$. Suppose that $m > r$. Then,

$$\begin{aligned}\left[\left(\mathcal{A}P^{-1}Q'\right)\left(\mathcal{A}P^{-1}Q'\right)'\right]^{(m-1)d+r,(m'-1)d+r'} &= \sum_{i=r+1}^{d}\alpha^{mi}(\alpha^{m'i}\alpha^{r'r} - \alpha^{m'r}\alpha^{r'i}) \\ &= \alpha^{m'm}\alpha^{r'r} - \alpha^{m'r}\alpha^{r'm} \\ &= \mathcal{Q}^{(m-1)d+r,(m'-1)d+r'}.\end{aligned}$$

Interchanging the roles between $r$ and $m$ above, we obtain the desired relation $\left(\mathcal{A}P^{-1}Q'\right)\left(\mathcal{A}P^{-1}Q'\right)' = \mathcal{Q}$, which completes our argument.

### 10.3 Proof of Proposition 3

First note that from Itô's lemma

$$\mathrm{d}\exp(X_s^k) = \exp(X_s^k)(\mathrm{d}X_s + \frac{1}{2}\mathrm{d}[X^k]_s)$$

$$\mathrm{d}\exp(Y_s^*) = \exp(Y_s^*)(\mathrm{d}Y_s^* + \frac{1}{2}\mathrm{d}[Y^*]_s),$$

and $\exp(Y_s^*) = \sum_{k=1}^{d} a_s^k \exp(X_s^k) = \sum_{k=1}^{d} w_s^k$. It thus follows that under the assumptions of Section 2, we have that

$$\mathrm{d}Y_s^* = \frac{1}{\exp(Y_s^*)}\sum_{k=1}^{d} w_s^k\left(\mathrm{d}X_s^k + \frac{1}{2}\mathrm{d}[X^k]_s\right) - \frac{1}{2}\mathrm{d}[Y^*]_s \tag{61}$$

$$[X^l, Y^*]_t = \sum_{k=1}^{d}\int_0^t \frac{w_s^k}{\exp(Y_s^*)}\mathrm{d}[X^k, X^l]_s. \tag{62}$$

For the weighted sum of betas

$$\sum_{l=1}^{d}\frac{w_s^l}{\exp(2Y_s^*)}\mathrm{d}\beta_s^l, \tag{63}$$

with $\beta_s^l$ as defined in (41) we have that from (62) the following result follows:

$$\sum_{l=1}^{d}\frac{w_s^l}{\exp(2Y_s^*)}\mathrm{d}\beta_s^l = \sum_{l=1}^{d}\frac{w_s^l}{\exp(Y_s^*)}\sum_{k=1}^{d}\frac{w_s^k}{\exp(Y_s^*)}\mathrm{d}[X^k, X^l]_s = \sum_{l=1}^{d}\frac{w_s^l}{\exp(Y_s^*)}\mathrm{d}[Y^*, X^l]_s = \mathrm{d}[Y^*]_s.$$

# 11 Appendix 2: Asymptotics for stochastic functionals of a localized HY estimator

Consider $k_n \in \mathbb{N}$ a window satisfying that $k_n \uparrow \infty$ and $k_n/n \to 0$, as $n \to \infty$ for a given Itô's semimartingale $H$ with representation

$$\begin{aligned} H_t =& H_0 + \int_0^t \mu'_s ds + \sum_{m=1}^{d'} \int_0^t \sigma'^m_s dB^m_s \\ &+ \int_0^t \int_E \varphi'(s,z) \mathbf{1}_{\|\varphi'(s,z)\| \le 1} (N-\lambda)(dsdz) + \int_0^t \int_E \varphi'(s,z) \mathbf{1}_{\|\varphi'(s,z)\|>1} N(dsdz), \end{aligned} \tag{64}$$

in which $\mu', \sigma'$ and $\delta'$ satisfy the same assumptions as $\tilde{\mu}$, $\tilde{\sigma}$ and $\delta$ in (3). Within this framework, we define

$$\psi^{kl}(H) = \int_0^1 H_s \Sigma^{kl}_s ds$$

and

$$\psi^{kl}_n(H) = \frac{1}{n_k} \sum_{m=1}^{n_k - k_n + 1} H_{t^k_{m-1}} \hat{\Sigma}^{kl}_{t^k_{m-1}}, \tag{65}$$

where

$$\hat{\Sigma}^{kl}_{t^k_m} = \frac{n_k}{k_n} \left\{ \bar{\Sigma}^{kl}_{t^k_m + k_n/n_k} - \bar{\Sigma}^{kl}_{t^k_m} \right\}, \quad m = 0, 1, \ldots, n_k - k_n. \tag{66}$$

For $H^{(1)}, \ldots, H^{(N)}$ processes of the form of (64) we use the notation

$$\Lambda(H^{(1)}, \ldots, H^{(m)}) = (\psi^{kl}_n(H^{(1)}) - \psi^{kl}(H^{(1)}), \ldots, \psi^{kl}_n(H^{(N)}) - \psi^{kl}(H^{(N)}))_{k,l=1,\ldots,d}.$$

We have the following result:

**Theorem 2.** *Assume that $X$ is given by (1) and let Assumption 1 hold. If $k_n^2/n \to 0$, then as $n \to \infty$*

$$\sqrt{n}\Lambda(H^{(1)}, \ldots, H^{(m)}) \stackrel{s.d}{\to} Z = (Z^{kl}_1, \ldots, Z^{kl}_N)_{k,l=1,\ldots,d},$$

*in which $Z$ is an $\mathcal{F}$-conditional centered Gaussian vector satisfying*

$$\mathbb{E}\left( Z^{kl}_x, Z^{k',l'}_y \middle| \mathcal{F} \right) = \int_0^1 H^{(x)}_s H^{(y)}_s \left( \Sigma^{k,k'}_s \Sigma^{l,l'}_s + \Sigma^{kl'}_s \Sigma^{k',l}_s \right) ds.$$

**Remark 2.** *Thanks to Lemma 4.4.9 in Jacod and Protter (2011), in all the proofs below we may and do assume* *that $\sigma, \mu, \tilde{\sigma}, \tilde{\mu}, \mu', \sigma'$ and $X$ are bounded as well as*

$$\|\varphi(\omega,t,z)\| + \left\|\varphi'(\omega,t,z)\right\| \leq \Gamma(z),$$

*where $\Gamma$ is a deterministic bounded measurable function fulfilling that $\int_E \Gamma(z)^2\nu(dz) < \infty$. Under these strong assumptions, we have that*

$$H_t = H_0 + \int_0^t \tilde{\mu}'_s ds + \sum_{m=1}^{d'} \int_0^t \sigma_s'^m dB_s^m + \int_0^t \int_E \varphi'(s,z)(N-\lambda)(dsdz),$$

*where $\tilde{\mu}'$ is bounded. Moreover, for all $s \geq 0$ and $p \geq 1$*

$$\begin{aligned}
\mathbb{E}\left( \sup_{u\leq s} \|X_{t+u} - X_t\|^p \middle| \mathcal{F}_t \right) &\leq C s^{p/2} \\
\|\mathbb{E}\left( X_{t+s} - X_t \middle| \mathcal{F}_s \right)\| &\leq Cs, \\
\mathbb{E}\left( \sup_{u\leq s} |\Sigma_{t+u} - \Sigma_t|^p \middle| \mathcal{F}_t \right) &\leq C s^{1\wedge p/2}, \\
\|\mathbb{E}\left( \Sigma_{t+s} - \Sigma_t \middle| \mathcal{F}_s \right)\| &\leq Cs, \\
\mathbb{E}\left( \sup_{u\leq s} |H_{t+u} - H_t|^p \middle| \mathcal{F}_t \right) &\leq C s^{1\wedge p/2}, \\
\|\mathbb{E}\left( H_{t+s} - H_t \middle| \mathcal{F}_s \right)\| &\leq Cs.
\end{aligned} \tag{67}$$

*For more details in this regard we refer to Section 2 in Jacod and Protter (2011).*

## 11.1 First approximation

For the remainder of this work, if $Z$ and $Y$ denote two stochastic processes, we will write

$$\begin{aligned}
\beta_p^{kl}(Y,Z) &= \sum_{i,j} \Delta_{i,k} Y \Delta_{j,l} Z \mathbf{1}_{(t_{p-1}^k, t_p^k]}(t_i^k \vee t_j^l) \mathbf{1}_{I_i^k \cap I_j^l \neq \emptyset}; \\
\tilde{\beta}_m^{kl}(Y,Z) &= \sum_{i,j} \Delta_{i,k} Y \Delta_{j,l} Z \mathbf{1}_{(\frac{m-1}{n}, \frac{m}{n}]}(t_i^k \vee t_j^l) \mathbf{1}_{I_i^k \cap I_j^l \neq \emptyset},
\end{aligned} \tag{68}$$

for $k = 1, \ldots, d, l = 1, \ldots, d'$, $p = 1, \ldots, n_k - 1$ and $m = 1, \ldots, n$. Note that

$$\psi^{kl}(H) = \sum_{p=1}^{n_k} h_{p,n} \beta_p^{kl}(X^k, X^l),$$

where

$$h_{p,n} = \frac{1}{k_n} \sum_{m=1\vee(p-k_n+1)}^{p\wedge(n_k-k_n+1)} H_{t_{m-1}^k}.$$

For the rest of this part we focus on showing the following approximation:

**Lemma 1.** *Assume that $X$ and $H$ are given by (1) and (64), respectively. Let Assumption 1 hold and suppose that $k_n^2/n \to 0$. Then,*

$$\begin{aligned}\psi^{kl}(H) =&\sum_{p=1}^{n_k} H_{t_p^k}\beta_p^{kl}(X^k, X^l) + \mathrm{o}_{\mathbb{P}}(n^{-1/2})\\ =&\sum_{i,j} H_{t_{i-1}^k \wedge t_{j-1}^l}\Delta_{i,k}X^k\Delta_{j,l}X^l\mathbf{1}_{I_i^k\cap I_j^l\neq\emptyset} + \mathrm{o}_{\mathbb{P}}(n^{-1/2})\\ =&\sum_{m=1}^{n} H_{\frac{m-1}{n}}\tilde{\beta}_m^{kl}(X^k, X^l) + \mathrm{o}_{\mathbb{P}}(n^{-1/2}).\end{aligned} \tag{69}$$

*Proof.* We will only show the first equality in (69). The other approximations can be shown using the same method. For $n$ sufficiently large, define the error of the first approximation by

$$R_{n,1} = \sum_{p=1}^{n_k}\left(h_{p,n} - H_{t_p^k}\right)\beta_p^{kl}(X^k, X^l).$$

In view that $I_i^k \cap I_j^l \neq \emptyset$ if and only if the following four scenarios occur

$$\begin{cases} t_i^k \geq t_j^l \geq t_{i-1}^k \text{ and } t_{i-1}^k \geq t_{j-1}^l\\ t_j^l \geq t_i^k \geq t_{j-1}^l \text{ and } t_{i-1}^k \geq t_{j-1}^l\\ t_i^k \geq t_j^l \geq t_{i-1}^k \text{ and } t_{j-1}^l \geq t_{i-1}^k\\ t_j^l \geq t_i^k \geq t_{j-1}^l \text{ and } t_{j-1}^l \geq t_{i-1}^k\end{cases}, \tag{70}$$

we deduce, from Assumption 1, the estimates in (67), Jensen's inequality and the Cauchy-Schwarz inequality that for all $J \geq 1$

$$\mathbb{E}\left(\left|\beta_p^{kl}(X^k, X^l)\right|^J\right) \leq C/n^J, \;\; p = 1, \ldots, n_k. \tag{71}$$

Thus, letting $l_n = k_n/n_k$ and using once again the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}\sum_{p=1}^{k_n}\mathbb{E}\left(\left|\left(h_{p,n} - H_{t_p^k}\right)\beta_p^{kl}(X^k, X^l)\right|\right) \leq& \frac{C}{n_k}\sum_{p=1}^{k_n}\mathbb{E}\left(\left|H_{t_p^k} - \frac{1}{k_n}\sum_{m=1}^{p} H_{t_{m-1}^k}\right|^2\right)^{1/2}\\ =&C\int_0^{l_n}\mathbb{E}\left(\left|H_{\frac{[sn_k]+1}{n_k}} - \frac{1}{l_n}\int_0^{\frac{[sn_k]+1}{n_k}} H_{\frac{[rn_k]}{n_k}}dr\right|^2\right)^{1/2} ds\\ =&Cl_n\int_0^1\mathbb{E}\left(\left|H_{\frac{[h_n rn_k]}{n_k}} - \int_0^{\frac{[sn_k]+1}{n_k}h_n} H_{\frac{[h_n sn_k]}{n_k}}ds\right|^2\right)^{1/2} dr \leq Cl_n.\end{aligned}$$

Similar calculations show that

$$\sum_{p=n_k-k_n+1}^{n_k} \mathbb{E}\left(\left|\left(h_{p,n} - H_{t_p^k}\right)\beta_p^{kl}(X^k, X^l)\right|\right) \le C l_n.$$

Thus, using that $\sqrt{n} l_n \to 0$, we conclude that

$$\begin{aligned}
R_{n,1} =& \frac{1}{k_n}\sum_{q=k_n+1}^{n_k-2k_n} \Delta_{q,k} H \sum_{p=q}^{q+k_n-1} \beta_p^{kl}(X^k, X^l)(q-p+k_n) \\
&+ \frac{1}{k_n}\sum_{q=1}^{k_n} \Delta_{q,k} H \sum_{p=k_n}^{q+k_n-1} \beta_p^{kl}(X^k, X^l)(q-p+k_n) \\
&+ \frac{1}{k_n}\sum_{q=n_k-2k_n+1}^{n_k-k_n-1} \Delta_{q,k} H \sum_{p=q}^{n_k-k_n} \beta_p^{kl}(X^k, X^l)(q-p+k_n) + \mathrm{o}_{\mathbb{P}}(n^{-1/2}).
\end{aligned}$$

Furthermore, in view that

$$\sum_{p=q}^{q+k_n-1} (q-p+k_n) = \mathrm{O}(k_n^2), \quad q = 1, \ldots, n_k - k_n, \tag{72}$$

we can apply (71) to deduce that

$$R_{n,1} = \frac{1}{k_n}\sum_{q=k_n+1}^{n_k-2k_n} \Delta_{q,k} H \sum_{p=q}^{q+k_n-1} \beta_p^{kl}(X^k, X^l)(q-p+k_n) + \mathrm{o}_{\mathbb{P}}(n^{-1/2}). \tag{73}$$

Now, let $A_r$ be the set of indexes $(i, j)$ satisfying the $r$th scenario in (70), put

$$\beta_{p,r}^{kl}(X^k, X^l) = \sum_{i,j \in A_r} \Delta_{i,k} X^k \Delta_{j,l} X^l \mathbf{1}_{(t_{p-1}^k, t_p^k]}(t_i^k \vee t_j^l) \mathbf{1}_{I_i^k \cap I_j^l \neq \emptyset}$$

and define

$$\begin{aligned}
U_{n,1}(r) =& \frac{\sqrt{n}}{k_n}\sum_{q=k_n+1}^{n_k-2k_n} \sum_{p=q+1}^{q+k_n-1} \Delta_{q,k} H \beta_{p,r}^{kl}(X^k, X^l)(q-p+k_n); \\
\tilde{U}_{n,1}(r) =& \sqrt{n}\sum_{q=k_n+1}^{n_k-2k_n} \Delta_{q,k} H \beta_{q,r}^{kl}(X^k, X^l).
\end{aligned}$$

Since $\sqrt{n} R_{n,1} = \sum_{r=1}^{4}\left(U_{n,1}(r) + \tilde{U}_{n,1}(r)\right)$, we need to show that $U_{n,1}(r)$ and $\tilde{U}_{n,1}(r)$ are asymptotically negligible for $r = 1, 2, 3, 4$. We will only show the case for $r = 1$. By Jensen's inequality and Lemma 2.2.12 in Jacod and Protter (2011), we need to show that

$$
\begin{aligned}
m_{n,U} &= \frac{\sqrt{n}}{k_n} \sum_{q=k_n+1}^{n_k-2k_n} \sum_{p=q+1}^{q+k_n-1} \mathbb{E}\left( \Delta_{q,k} H \beta^{kl}_{p,1}(X^k, X^l) \middle| \mathcal{F}_{t^k_{q-1}} \right) (q-p+k_n) = \mathrm{o}_{\mathbb{P}}(1), \\
m_{n,\tilde{U}} &= \sqrt{n} \sum_{q=k_n+1}^{n_k-2k_n} \mathbb{E}\left( \Delta_{q,k} H \beta^{kl}_{q,1}(X^k, X^l) \middle| \mathcal{F}_{t^k_{q-1}} \right) = \mathrm{o}_{\mathbb{P}}(1)
\end{aligned}
\tag{74}
$$

and

$$
\begin{aligned}
v_{n,U} &= \frac{n}{k_n^2} \sum_{q=k_n+1}^{n_k-2k_n} \mathbb{E}\left[ \left( \Delta_{q,k} H \sum_{p=q+1}^{q+k_n-1} \beta^{kl}_{p,1}(X^k, X^l)(q-p+k_n) \right)^2 \middle| \mathcal{F}_{t^k_{q-1}} \right] = \mathrm{o}_{\mathbb{P}}(1), \\
v_{n,\tilde{U}} &= n \sum_{q=k_n+1}^{n_k-2k_n} \mathbb{E}\left[ \left( \Delta_{q,k} H \beta^{kl}_{q,1}(X^k, X^l) \right)^2 \middle| \mathcal{F}_{t^k_{q-1}} \right] = \mathrm{o}_{\mathbb{P}}(1).
\end{aligned}
\tag{75}
$$

From Remark 2 and Itô's formula, we have that for any $t \geq s \geq u$ and $k, l = 1, \dots, d$

$$
\begin{aligned}
\mathbb{E}\left[ (H_t - H_s)\left(X^k_t - X^k_s\right) \middle| \mathcal{F}_u \right] =& \mathbb{E}\left[ \int_s^t (H_r - H_s)\mu^k_r dr \middle| \mathcal{F}_u \right] + \mathbb{E}\left[ \int_s^t (X^k_r - X^k_s)\tilde{\mu}'_r dr \middle| \mathcal{F}_u \right] \\
&+ \mathbb{E}\left[ \int_s^t \varphi^{(1)}_r dr \middle| \mathcal{F}_u \right].
\end{aligned}
\tag{76}
$$

We also have that

$$
\begin{aligned}
\mathbb{E}\left[ (H_t - H_s)\left(X^k_t - X^k_s\right)\left(X^l_t - X^l_s\right) \middle| \mathcal{F}_u \right] =& \mathbb{E}\left[ \int_s^t (H_r - H_s)\left(X^k_r - X^k_s\right)\mu^l_r dr \middle| \mathcal{F}_u \right] \\
&+ \mathbb{E}\left[ \int_s^t (H_r - H_s)\left(X^l_r - X^l_s\right)\mu^k_r dr \middle| \mathcal{F}_u \right] \\
&+ \mathbb{E}\left[ \int_s^t \left(X^k_r - X^k_s\right)\left(X^l_r - X^l_s\right)\tilde{\mu}'_r dr \middle| \mathcal{F}_u \right] \\
&+ \mathbb{E}\left[ \int_s^t \left(X^k_r - X^k_s\right)\varphi^{(2)}_r dr \middle| \mathcal{F}_u \right] \\
&+ \mathbb{E}\left[ \int_s^t \left(X^l_r - X^l_s\right)\varphi^{(1)}_r dr \middle| \mathcal{F}_u \right] + \mathbb{E}\left[ \int_s^t (H_r - H_s)\,\Sigma^{kl}_r dr \middle| \mathcal{F}_u \right],
\end{aligned}
\tag{77}
$$

for some càdlàg processes $\varphi^{(1)}, \varphi^{(2)}$ depending only on $\sigma$ and $\sigma'$. Let $q \in \{k_n+1, \dots, n_k - 2k_n\}$. It follows from (76), (77), Assumption 1 and (67) that if $(p, j) \in A_1$ is such that $t^l_{j-1} \geq t^k_q$ and $p \geq q+1$, we have as in (76) that

$$
\left| \mathbb{E}\left( \Delta_{p,k} X^k \Delta_{j,l} X^l \middle| \mathcal{F}_{t^k_q} \right) - (t^l_j - t^k_{p-1})\Sigma^{kl}_{t^k_{q-1}} \right| \leq C n^{-3/2},
\tag{78}
$$

while for $t^k_q \geq t^l_{j-1}$

$$\begin{aligned}\mathbb{E}\left(\Delta_{p,k}X^k\Delta_{j,l}X^l\middle|\mathcal{F}_{t^k_q}\right) &= \left(X^l_{t^k_q} - X^l_{t^l_{j-1}}\right)\mathbb{E}\left(\Delta_{p,k}X^k\middle|\mathcal{F}_{t^k_q}\right)\\ &\quad + \mathbb{E}\left(\Delta_{p,k}X^k\left(X^l_{t^l_j} - X^l_{t^k_q}\right)\middle|\mathcal{F}_{t^k_q}\right)\\ &= \left(X^l_{t^k_q} - X^l_{t^l_{j-1}}\right)\int_{t^k_{p-1}}^{t^k_p}\mathbb{E}\left(\mu^k_r\middle|\mathcal{F}_{t^k_q}\right)dr\\ &\quad + (t^l_j - t^k_{p-1})\Sigma^{kl}_{t^k_{q-1}}\\ &\quad + \mathrm{O}_{\mathbb{P}}(n^{-3/2}).\end{aligned}$$

We conclude that

$$\mathbb{E}\left[\left|\mathbb{E}\left(\Delta_{q,k}H\Delta_{p,k}X^k\Delta_{j,l}X^l\middle|\mathcal{F}_{t^k_{q-1}}\right) - (t^l_j - t^k_{p-1})\Sigma^{kl}_{t^k_{q-1}}\mathbb{E}\left(\Delta_{q,k}H\middle|\mathcal{F}_{t^k_{q-1}}\right)\right|\right] \leq Cn^{-2}.$$

Therefore,

$$m_{n,U} = \frac{\sqrt{n}}{k_n}\sum_{q=k_n+1}^{n_k-2k_n}\sum_{p=q+1}^{q+k_n-1}\sum_j(t^l_j - t^k_{p-1})\Sigma^{kl}_{t^k_{q-1}}\mathbb{E}\left(\Delta_{q,k}H\middle|\mathcal{F}_{t^k_{q-1}}\right)\mathbf{1}_{(p,j)\in A_1}(q-p+k_n) + \mathrm{o}_{\mathbb{P}}(1) = \mathrm{o}_{\mathbb{P}}(1),$$

where we have also used (72) and (67). To deal with $m_{n,\tilde{U}}$ first note that

$$\begin{aligned}\mathbb{E}\left(\Delta_{q,k}H\Delta_{q,k}X^k\Delta_{j,l}X^l\middle|\mathcal{F}_{t^k_{q-1}}\right) =& (X^l_{t^k_{q-1}} - X^l_{t^l_{j-1}})\times\mathbb{E}\left(\Delta_{q,k}H\Delta_{q,k}X^k\middle|\mathcal{F}_{t^k_{q-1}}\right)\\ &+ \mathbb{E}\left(\left(H_{t^l_j} - H_{t^k_{q-1}}\right)\left(X^k_{t^l_j} - X^k_{t^k_{q-1}}\right)\left(X^l_{t^l_j} - X^l_{t^k_{q-1}}\right)\middle|\mathcal{F}_{t^k_{q-1}}\right)\\ &+ \mathbb{E}\left(\left(H_{t^k_q} - H_{t^l_j}\right)\left(X^k_{t^k_j} - X^k_{t^k_{q-1}}\right)\left(X^l_{t^l_j} - X^l_{t^k_{q-1}}\right)\middle|\mathcal{F}_{t^k_{q-1}}\right)\\ &+ \mathbb{E}\left(\left(H_{t^l_j} - H_{t^k_{q-1}}\right)\left(X^k_{t^k_q} - X^k_{t^k_j}\right)\left(X^l_{t^l_j} - X^l_{t^k_{q-1}}\right)\middle|\mathcal{F}_{t^k_{q-1}}\right)\\ &+ \mathbb{E}\left(\left(H_{t^k_q} - H_{t^l_j}\right)\left(X^k_{t^k_q} - X^k_{t^k_j}\right)\left(X^l_{t^l_j} - X^l_{t^k_{q-1}}\right)\middle|\mathcal{F}_{t^k_{q-1}}\right)\\ =& \mathrm{O}_{\mathbb{P}}(n^{-2}) + \mathbb{E}\left[\int_{t^k_{q-1}}^{t^k_q}\left(X^l_{t^k_{q-1}} - X^l_{t^k_{j-1}}\right)\varphi^{(1)}_r dr\middle|\mathcal{F}_{t^k_{q-1}}\right]\\ &+ \mathbb{E}\left[\int_{t^k_{q-1}}^{t^l_j}\left(X^k_r - X^k_{t^k_{q-1}}\right)\varphi^{(2)}_r dr\middle|\mathcal{F}_{t^k_{q-1}}\right] + \mathbb{E}\left[\int_{t^k_{q-1}}^{t^k_j}\left(X^l_r - X^l_{t^k_{q-1}}\right)\varphi^{(1)}_r dr\middle|\mathcal{F}_{t^k_{q-1}}\right]\\ &\mathbb{E}\left[\int_{t^l_j}^{t^k_q}\left(X^l_{t^k_j} - X^l_{t^k_{q-1}}\right)\varphi^{(1)}_r dr\middle|\mathcal{F}_{t^k_{q-1}}\right];\end{aligned}$$

so

$$
\begin{aligned}
m_{n,\tilde{U}} =& \sqrt{n}\sum_{q=k_n+1}^{n_k-2k_n}\sum_{j}\mathbb{E}\left[\int_{t^k_{q-1}}^{t^k_q}\left(X^l_{t^k_{q-1}}-X^l_{t^k_{j-1}}\right)\varphi^{(1)}_r dr\middle|\mathcal{F}_{t^k_{q-1}}\right]\mathbf{1}_{(q,j)\in A_1}\\
&+\sqrt{n}\sum_{q=k_n+1}^{n_k-2k_n}\sum_{j}\mathbb{E}\left[\int_{t^k_{q-1}}^{t^l_j}\left(X^k_r-X^k_{t^k_{q-1}}\right)\varphi^{(2)}_r dr\middle|\mathcal{F}_{t^k_{q-1}}\right]\mathbf{1}_{(q,j)\in A_1}\\
&+\sqrt{n}\sum_{q=k_n+1}^{n_k-2k_n}\mathbb{E}\left[\int_{t^k_{q-1}}^{t^k_j}\left(X^l_r-X^l_{t^k_{q-1}}\right)\varphi^{(1)}_r dr\middle|\mathcal{F}_{t^k_{q-1}}\right]\mathbf{1}_{(q,j)\in A_1}\\
&+\sqrt{n}\sum_{q=k_n+1}^{n_k-2k_n}\mathbb{E}\left[\int_{t^l_j}^{t^k_q}\left(X^l_{t^k_j}-X^l_{t^k_{q-1}}\right)\varphi^{(1)}_r dr\middle|\mathcal{F}_{t^k_{q-1}}\right]\mathbf{1}_{(q,j)\in A_1}\\
&+\mathrm{o}_{\mathbb{P}}(1).
\end{aligned}
\tag{79}
$$

Since $\varphi^{(1)}$ is càdlàg, the Cauchy-Schwarz inequality and the Dominated Convergence Theorem guarantee that the sum

$$
\sqrt{n}\sum_{q=k_n+1}^{n_k-2k_n}\sum_{j}\mathbb{E}\left\{\left|\mathbb{E}\left[\int_{t^k_{q-1}}^{t^k_q}\left(X^l_{t^k_{q-1}}-X^l_{t^k_{j-1}}\right)\left(\varphi^{(1)}_r-\varphi^{(1)}_{t^k_{q-1}}\right)dr\middle|\mathcal{F}_{t^k_{q-1}}\right]\right|\right\}\mathbf{1}_{(q,j)\in A_1}
$$

is bounded (up to a constant) by

$$
\sum_{q=k_n+1}^{n_k-2k_n}\int_{t^k_{q-1}}^{t^k_q}\mathbb{E}\left(\left\|\varphi^{(1)}_r-\varphi^{(1)}_{t^k_{q-1}}\right\|^2\right)^{1/2}dr\to 0. \tag{80}
$$

Consequently, the first term in (79) equals

$$
\sqrt{n}\sum_{q=k_n+1}^{n_k-2k_n}\sum_{j}\varphi^{(1)}_{t^k_{q-1}}\int_{t^k_{q-1}}^{t^k_q}\mathbb{E}\left[\left(X^l_{t^k_{q-1}}-X^l_{t^k_{j-1}}\right)\middle|\mathcal{F}_{t^k_{q-1}}\right]dr\mathbf{1}_{(q,j)\in A_1}+\mathrm{o}_{\mathbb{P}}(1)=\mathrm{o}_{\mathbb{P}}(1), \tag{81}
$$

thanks to (67). A similar argument can be applied to the other summands in (79) in order to deduce that (74) is indeed true. Now we concentrate on showing that (75) is satisfied. Fix $q\in\{k_n+1,\ldots,n_k-2k_n\}$ and pick $(p,j),(p',j')\in A_1$ such that $q+k_n-1\geq p,p'\geq q+1$ and put

$$
\alpha^{kl}_{p,p',j,j'}(q)=\mathbb{E}\left(\Delta_{p,k}X^k\Delta_{p',k}X^k\Delta_{j,l}X^l\Delta_{j',l}X^l\middle|\mathcal{F}_{t^k_q}\right).
$$

Suppose first that $j' \geq j$. Then,

$$\alpha^{kl}_{p,p',j,j'}(q) = \begin{cases} \mathrm{O}(n^2) & \text{if } t^l_{j-1} \geq t^k_q; \\ \left(X^l_{t^k_q} - X^l_{t^l_{j-1}}\right) \mathbb{E}\left(\Delta_{p,k}X^k \Delta_{p',k}X^k \Delta_{j',l}X^l \middle| \mathcal{F}_{t^k_q}\right) + \mathrm{O}(n^{-2}) & \text{if } t^l_{j'-1} \geq t^k_q > t^l_{j-1}; \\ \left(X^l_{t^k_q} - X^l_{t^l_{j-1}}\right)\left(X^l_{t^k_q} - X^l_{t^l_{j'-1}}\right) \mathbb{E}\left(\Delta_{p,k}X^k \Delta_{p',k}X^k \middle| \mathcal{F}_{t^k_q}\right) + \mathrm{O}(n^{-2}) & t^k_q > t^l_{j'-1} \geq t^l_{j-1}. \end{cases}$$

Moreover, thanks to (76) and (78), if $t^l_{j'-1} \geq t^k_q > t^l_{j-1}$,

$$\left|\mathbb{E}\left(\Delta_{p,k}X^k \Delta_{p',k}X^k \Delta_{j',l}X^l \middle| \mathcal{F}_{t^k_q}\right)\right| \leq \begin{cases} C/n^2 & \text{if } t^k_p > t^k_{p'} \text{ or if } t^l_{j'-1} \geq t^k_p; \\ C/n^{3/2} & t^k_{p'} \geq t^k_p > t^l_{j'-1}, \end{cases}$$

as well as

$$\left|\mathbb{E}\left(\Delta_{p,k}X^k \Delta_{p',k}X^k \middle| \mathcal{F}_{t^k_q}\right)\right| \leq \begin{cases} C/n^2 & \text{if } p \neq p'; \\ C/n & \text{if } p = p. \end{cases}$$

Hence, if $j \geq j'$,

$$\mathbb{E}\left\{\left|\mathbb{E}\left[(\Delta_{q,k}H)^2 \Delta_{p,k}X^k \Delta_{j,l}X^l \Delta_{p',k}X^k \Delta_{j',l}X^l \middle| \mathcal{F}_{t^k_{q-1}}\right]\right|\right\} \leq Cn^{-5/2},$$

whenever $t^l_{j'-1} \geq t^k_q > t^l_{j-1}$ and $t^k_{p'} \geq t^k_p > t^l_{j'-1}$ or when $t^k_q > t^l_{j'-1} \geq t^l_{j-1}$ and $p \neq p'$. Otherwise, we have that

$$\mathbb{E}\left\{\left|\mathbb{E}\left[(\Delta_{q,k}H)^2 \Delta_{p,k}X^k \Delta_{j,l}X^l \Delta_{p',k}X^k \Delta_{j',l}X^l \middle| \mathcal{F}_{t^k_{q-1}}\right]\right|\right\} \leq Cn^{-3}.$$

Interchanging $j$ with $j'$ above and applying (72), we can conclude that

$$\begin{aligned} \mathbb{E}\left(|v_{n,U}|\right) \leq & C\frac{1}{n^{3/2}} \sum_{q=k_n+1}^{n_k-2k_n} \sum_{p,p'=q+1}^{q+k_n-1} \sum_{j' \geq j} \mathbf{1}_{t^l_{j'-1} \geq t^k_q > t^l_{j-1}} \mathbf{1}_{(p,j),(p',j') \in A_1} \mathbf{1}_{t^k_{p'} \geq t^k_p > t^l_{j'-1}} \\ & + \mathrm{o}(1). \end{aligned}$$

The first part of (75) now follows from the fact that the last sum contains $\mathrm{O}(nk_n)$ terms due to Assumption 1. Finally, from the estimates in (67), the Cauchy-Schwarz inequality and (71) we obtain that

$$\mathbb{E}\left[\left(\Delta_{q,k}H\beta^{kl}_{q,1}(X^k, X^l)\right)^2\right] \leq C/n^{5/2},$$

which trivially implies that $v_{n,\tilde{U}}$ is negligible as $n \to \infty$. This concludes our argument.

□

## 11.2 Negligibility of the drift component

From Lemma 1, we have that

$$\begin{aligned} \psi^{kl}(H) = &\sum_{i,j} H_{t^k_{i-1}\wedge t^l_{j-1}} \Delta_{i,k} M^k \Delta_{j,l} M^l \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} + \mathrm{o}_{\mathbb{P}}(n^{-1/2}) \\ &+ \sum_{i,j} H_{t^k_{i-1}\wedge t^l_{j-1}} \left( \Delta_{i,k} M^k \Delta_{j,l} A^l + \Delta_{i,k} A^k \Delta_{j,l} M^l + \Delta_{i,k} A^k \Delta_{j,l} A^l \right) \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} \end{aligned}, \tag{82}$$

whenever $k_n^2/n \to 0$. For the rest of this subsection we will show that under our framework the last summand in (82) is $\mathrm{o}_{\mathbb{P}}\left(n^{-1/2}\right)$. From Assumption 1, it follows easily that

$$\sum_{i,j} \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} = \mathrm{O}\left(n\right).$$

From this, the boundedness of $H$ and the fact that $\left|\Delta_{i,k} A^k \Delta_{j,l} A^l\right| \leq C/n^2$ , we easily deduce that

$$\sum_{i,j} H_{t^k_{i-1}\wedge t^l_{j-1}} \Delta_{i,k} A^k \Delta_{j,l} A^l \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} = \mathrm{o}_{\mathbb{P}}\left(n^{-1/2}\right).$$

On the other hand, reasoning as in (80), let us conclude that

$$\sum_{i,j} H_{t^k_{i-1}\wedge t^l_{j-1}} \Delta_{i,k} A^k \Delta_{j,l} M^l \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} = \frac{1}{n_k} \sum_{i,j} H_{t^k_{i-1}\wedge t^l_{j-1}} \mu^k_{t^k_{i-1}\wedge t^l_{j-1}} \Delta_{j,l} M^l \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} + \mathrm{o}_{\mathbb{P}}\left(n^{-1/2}\right).$$

Hence, in view that $\mu^k_{t^k_{i-1}\wedge t^l_{j-1}} H_{t^k_{i-1}\wedge t^l_{j-1}}$ is $\mathcal{F}_{t^l_{j-1}}$-measurable and bounded, $\mathbb{E}\left(\Delta_{j,k} M^l \middle| \mathcal{F}_{t^l_{j-1}}\right) = 0$, and $\mathbb{E}\left(\left|\Delta_{j,k} M^l\right|^2 \middle| \mathcal{F}_{t}\right.$
$C/n$, it follows from Lemma 2.2.11 in Jacod and Protter (2011) that

$$\sum_{i,j} H_{t^k_{i-1}\wedge t^l_{j-1}} \Delta_{i,k} A^k \Delta_{j,l} M^l \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} = \mathrm{o}_{\mathbb{P}}\left(n^{-1/2}\right),$$

as claimed. The negligibility of the sum involving $\Delta_{i,k} M^k \Delta_{j,l} A^l$ can be shown in the same way.

## 11.3 Freezing the volatility

In this part we show that under our set-up we can replace $\Delta_{i,k} M^k$ with $\sum_x \sigma^{k,x}_{t^k_{i-1}\wedge t^l_{j-1}} \Delta_{i,k} B^x$ in

$$S^{kl}_n = \sum_{i,j} H_{t^k_{i-1}\wedge t^l_{j-1}} \Delta_{i,k} M^k \Delta_{j,l} M^l \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset}.$$

**Lemma 2.** *Assume that $H$ is given by (64) and let Assumption 1 hold. Then,*

$$S_n^{kl} = \sum_{x,y}\sum_{i,j} H_{t_{i-1}^k \wedge t_{j-1}^l} \sigma^{k,x}_{t_{i-1}^k \wedge t_{j-1}^l} \sigma^{l,y}_{t_{i-1}^k \wedge t_{j-1}^l} \chi_{i,j}^{kl} \mathbf{1}_{I_i^k \cap I_j^l \neq \emptyset} + \mathrm{o}_{\mathbb{P}}\left(n^{-1/2}\right), \tag{83}$$

*where*

$$\chi_{i,j}^{kl} = \Delta_{i,k} B^x \Delta_{j,l} B^y.$$

*Proof.* We have that

$$\begin{aligned} S_n^{kl} =& \sum_{i,j} H_{t_{i-1}^k \wedge t_{j-1}^l} \chi_{i,j}^{kl} \mathbf{1}_{I_i^k \cap I_j^l \neq \emptyset}, \\ &+ \sum_{r=1}^{3} \sum_{i,j} H_{t_{i-1}^k \wedge t_{j-1}^l} \chi_{i,j}^{kl}(r), \end{aligned}$$

where

$$\begin{aligned} \chi_{i,j}^{kl}(1) =& \sum_{x,y} \int_{t_{i-1}^k}^{t_i^k} \left(\sigma_s^{k,x} - \sigma^{k,x}_{t_{i-1}^k \wedge t_{j-1}^l}\right) dB_s^x \int_{t_{j-1}^k}^{t_j^l} \left(\sigma_s^{l,y} - \sigma^{k,y}_{t_{i-1}^k \wedge t_{j-1}^l}\right) dB_s^y; \\ \chi_{i,j}^{kl}(2) =& \sum_{x,y} \int_{t_{i-1}^k}^{t_i^k} \left(\sigma_s^{k,x} - \sigma^{k,x}_{t_{i-1}^k \wedge t_{j-1}^l}\right) dB_s^x \sigma^{l,y}_{t_{i-1}^k \wedge t_{j-1}^l} \Delta_{j,l} B^y; \\ \chi_{i,j}^{kl}(3) =& \sum_{x,y} \int_{t_{j-1}^k}^{t_j^l} \left(\sigma_s^{l,y} - \sigma^{k,y}_{t_{i-1}^k \wedge t_{j-1}^l}\right) dB_s^y \sigma^{k,x}_{t_{i-1}^k \wedge t_{j-1}^l} \Delta_{i,k} B^x. \end{aligned}$$

The Cauchy-Swartz inequality and (67) imply that

$$\mathbb{E}\left(\left|\chi_{i,j}^{kl}(1)\right|\right) \leq C/n^2,$$

so $\sum_{i,j} H_{t_{i-1}^k \wedge t_{j-1}^l} \chi_{i,j}^{kl}(1) = \mathrm{o}_{\mathbb{P}}(n^{-1/2})$. Under the notation of the proof of Lemma 1, it is left to show that

$$E_{n,v}(r) = \sum_{i,j \in A_v} H_{t_{i-1}^k \wedge t_{j-1}^l} \chi_{i,j}^{kl}(r) = \mathrm{o}_{\mathbb{P}}(n^{-1/2}), \;\; v = 1, \ldots, 4, r = 2, 3.$$

We concentrate only on $E_{n,1}(2)$, since the other terms require similar calculations. If $(i,j)\in A_1$, then

$$\begin{aligned}
\chi^{kl}_{i,j}(2)=&\sum_{x,y}\int_{t^l_j}^{t^k_i}\left(\sigma^{k,x}_s-\sigma^{k,x}_{t^l_{j-1}}\right)dB^x_s\sigma^{l,y}_{t^l_{j-1}}\int_{t^l_{j-1}}^{t^l_j}dB^y_s\\
&\sum_{x,y}\int_{t^k_{i-1}}^{t^l_j}\left(\sigma^{k,x}_s-\sigma^{k,x}_{t^l_{j-1}}\right)dB^x_s\sigma^{l,y}_{t^l_{j-1}}\int_{t^k_{i-1}}^{t^l_j}dB^y_s\\
&\sum_{x,y}\int_{t^k_{i-1}}^{t^l_j}\left(\sigma^{k,x}_s-\sigma^{k,x}_{t^l_{j-1}}\right)dB^x_s\sigma^{l,y}_{t^l_{j-1}}\int_{t^l_{j-1}}^{t^k_{i-1}}dB^y_s\\
=&\sum_{w=1}^{3}\chi^{kl}_{i,j}(2,w).
\end{aligned}$$

We start by decomposing

$$\begin{aligned}
\int_{t^l_j}^{t^k_i}\left(\sigma^{k,x}_s-\sigma^{k,x}_{t^l_{j-1}}\right)dB^x_s\sigma^{l,y}_{t^l_{j-1}}\int_{t^l_{j-1}}^{t^l_j}\Delta_{j,l}B^y=&\sigma^{l,y}_{t^l_{j-1}}\int_{t^l_j}^{t^k_i}\left(\sigma^{k,x}_s-\sigma^{k,x}_{t^l_{j-1}}\right)dB^x_s\int_{t^l_{j-1}}^{t^k_{i-1}}dB^y_s\\
&+\sigma^{l,y}_{t^l_{j-1}}\int_{t^l_j}^{t^k_i}\left(\sigma^{k,x}_s-\sigma^{k,x}_{t^l_{j-1}}\right)dB^x_s\int_{t^k_{i-1}}^{t^l_j}dB^y_s\\
=:&\gamma^{x,y}_{i,j}(1)+\gamma^{x,y}_{i,j}(2).
\end{aligned}$$

Thus, for $J=1,2$, $\mathbb{E}\left(\gamma^{x,y}_{i,j}(J)\middle|\mathcal{F}_{t^k_{i-1}}\right)=0$ and

$$\mathbb{E}\left(\gamma^{x,y}_{i,j}(J)^2\right)\leq\frac{C}{n}\int_{t^k_{i-1}}^{t^k_i}\mathbb{E}\left[\left\|\sigma_s-\sigma_{t^l_{j-1}}\right\|^4\right]^{1/2}ds.$$

Since $\sigma$ is càdlàg, we can argue as in (80) and obtain that

$$\sum_{i,j\in A_1}H_{t^k_{i-1}\wedge t^l_{j-1}}\chi^{kl}_{i,j}(1,1)=\mathrm{o}_{\mathbb{P}}(n^{-1/2}).$$

Analogous reasoning it follows also that $\sum_{i,j\in A_1}H_{t^k_{i-1}\wedge t^l_{j-1}}\chi^{kl}_{i,j}(1,3)=\mathrm{o}_{\mathbb{P}}(n^{-1/2})$. For $\chi^{kl}_{i,j}(1,2)$ we first note that it is $\mathcal{F}_{t^l_j}$-measurable, so from (67)

$$\begin{aligned}
\mathbb{E}\left(\int_{t^k_{i-1}}^{t^l_j}\left(\sigma^{k,x}_s-\sigma^{k,x}_{t^l_{j-1}}\right)dB^x_s\int_{t^k_{i-1}}^{t^l_j}\Delta_{j,l}B^y\middle|\mathcal{F}_{t^l_{j-1}}\right)=&\int_{t^k_{i-1}}^{t^l_j}\mathbb{E}\left(\left(\sigma^{k,x}_s-\sigma^{k,x}_{t^k_{i-1}}\right)\middle|\mathcal{F}_{t^l_{j-1}}\right)ds\delta^{x,y}\\
=&\mathrm{O}(n^{-2}).
\end{aligned}$$

where $\delta^{x,y}$ denotes the Dirac's delta measure. Thus,

$$\sum_{i,j\in A_1} H_{t^k_{i-1}\wedge t^l_{j-1}} \mathbb{E}\left( \chi^{kl}_{i,j}(1,2) \middle| \mathcal{F}_{t^l_{j-1}} \right) = \mathrm{o}_{\mathbb{P}}(n^{-1/2}).$$

Finally, by Itô's formula,

$$\begin{aligned}
\mathbb{E}\left[ \left( \int_{t^k_{i-1}}^{t^l_j} \left( \sigma^{k,x}_s - \sigma^{k,x}_{t^l_{j-1}} \right) dB^x_s \int_{t^k_{i-1}}^{t^l_j} \Delta_{j,l} B^y \right)^2 \middle| \mathcal{F}_{t^l_{j-1}} \right] &= \int_{t^k_{i-1}}^{t^l_j} \mathbb{E}\left[ \left( \int_{t^k_{i-1}}^{s} dB^y_r \right)^2 \left( \sigma^{k,x}_s - \sigma^{k,x}_{t^l_{j-1}} \right)^2 \middle| \mathcal{F}_{t^l_{j-1}} \right] ds \\
&\quad + \int_{t^k_{i-1}}^{t^l_j} \int_{t^k_{i-1}}^{s} \mathbb{E}\left[ \left( \sigma^{k,x}_r - \sigma^{k,x}_{t^l_{j-1}} \right)^2 \middle| \mathcal{F}_{t^l_{j-1}} \right] dr ds \\
&\quad + 4 \int_{t^k_{i-1}}^{t^l_j} \mathbb{E}\left[ \int_{t^k_{i-1}}^{s} \left( \sigma^{k,x}_r - \sigma^{k,x}_{t^l_{j-1}} \right) dB^x_r \int_{t^k_{i-1}}^{s} \Delta_{j,l} B^y_r \middle| \mathcal{F}_{t^l_{j-1}} \right] ds \delta^{x,y} \\
&= \int_{t^k_{i-1}}^{t^l_j} \mathbb{E}\left[ \left( \int_{t^k_{i-1}}^{s} dB^y_r \right)^2 \left( \sigma^{k,x}_s - \sigma^{k,x}_{t^l_{j-1}} \right)^2 \middle| \mathcal{F}_{t^l_{j-1}} \right] ds \\
&\quad + \mathrm{O}(n^{-3}).
\end{aligned}$$

Hence,

$$\mathbb{E}\left\{ \left| \mathbb{E}\left[ \left( \int_{t^k_{i-1}}^{t^l_j} \left( \sigma^{k,x}_s - \sigma^{k,x}_{t^l_{j-1}} \right) dB^x_s \int_{t^k_{i-1}}^{t^l_j} \Delta_{j,l} B^y \right)^2 \middle| \mathcal{F}_{t^l_{j-1}} \right] \right| \right\} \leq \frac{C}{n} \int_{t^l_{j-1}}^{t^l_j} \mathbb{E}\left[ \left\| \sigma_s - \sigma_{t^l_{j-1}} \right\|^4 \right]^{1/2} ds + \mathrm{O}(n^{-3}),$$

which, as above, implies that

$$\sum_{i,j\in A_1} H_{t^k_{i-1}\wedge t^l_{j-1}} \chi^{kl}_{i,j}(1,2) = \mathrm{o}_{\mathbb{P}}(n^{-1/2}).$$

□

## 11.4 Proof for Theorem 2

The proof resembles the proof of Theorem 3.4 in Christensen et al. (2013). Let us start by stating some consequences of the previous subsections: From Lemmas 1 and 2 and the arguments therein, we have that if $k_n^2/n \to 0$ as $n \to \infty$, then for $k, l, k', l' = 1, \ldots, d$ and $u, v = 1, \ldots, N$

$$\psi^{kl}(H^{(u)}) = \sum_{i,j} \sum_{x,y} Y^{(u),k,x,l,y}_{t^k_{i-1}\wedge t^l_{j-1}} \Delta_{i,k} B^x \Delta_{j,l} B^y \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} + \mathrm{o}_{\mathbb{P}}(n^{-1/2}),$$

where $Y^{(u),k,x,l,y}_t = H^{(u)}_t \sigma^{k,x}_t \sigma^{l,y}_t$. In order to deal properly with measurability issues and correlation structure, we will make use of the big and small blocks technique introduced in Jacod et al. (2009). Thus we introduce the following objects: Let $\Theta > 0$ and $n_0 \in \mathbb{N}$ such that $\Theta > \max_{k=1,\ldots,d}(n/n_k)$ for all $n \geq n_0$. For $p > 0$ and

$m = 1, \ldots, \left[\frac{n}{p+\Theta}\right]$, put

$$I^{(1)}_{m,\Theta}(p) = \left[\frac{(m-1)(p+\Theta)}{n}, \frac{(m-1)(p+\Theta)+p}{n}\right);$$
$$I^{(2)}_{m,\Theta}(p) = \left[\frac{(m-1)(p+\Theta)+p}{n}, \frac{m(p+\Theta)}{n}\right),$$

and $I_{m,\Theta}(p) = I^{(1)}_{m,\Theta}(p) \cup I^{(2)}_{m,\Theta}(p)$. Using this notation, it is easy to see that for $p > \Theta$ and $n \geq n_0$ it holds that

$$\begin{aligned}
\sum_{i,j} a_{i,j} \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} = & \sum_{m=1}^{[n/(p+\Theta)]} \sum_{i,j} a_{i,j} \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} \mathbf{1}_{I^{(1)}_{m,\Theta}(p)}(t^k_{i-1}) \mathbf{1}_{I^{(1)}_{m,\Theta}(p)}(t^l_{j-1}) \\
& + \sum_{m=1}^{[n/(p+\Theta)]} \sum_{i,j} a_{i,j} \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} \mathbf{1}_{I^{(2)}_{m,\Theta}(p)}(t^k_{i-1}) \mathbf{1}_{I^{(2)}_{m,\Theta}(p)}(t^l_{j-1}) \\
& + \sum_{m=1}^{[n/(p+\Theta)]} \sum_{i,j} a_{i,j} \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} \mathbf{1}_{I^{(1)}_{m,\Theta}(p)}(t^k_{i-1}) \mathbf{1}_{I^{(2)}_{m,\Theta}(p)}(t^l_{j-1}) \\
& + \sum_{m=1}^{[n/(p+\Theta)]} \sum_{i,j} a_{i,j} \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} \mathbf{1}_{I^{(1)}_{m,\Theta}(p)}(t^k_{i-1}) \mathbf{1}_{I^{(2)}_{m-1,\Theta}(p)}(t^l_{j-1}) \\
& + \sum_{m=1}^{[n/(p+\Theta)]} \sum_{i,j} a_{i,j} \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} \mathbf{1}_{I^{(2)}_{m,\Theta}(p)}(t^k_{i-1}) \mathbf{1}_{I^{(1)}_{m,\Theta}(p)}(t^l_{j-1}) \\
& + \sum_{m=1}^{[n/(p+\Theta)]} \sum_{i,j} a_{i,j} \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} \mathbf{1}_{I^{(2)}_{m,\Theta}(p)}(t^k_{i-1}) \mathbf{1}_{I^{(1)}_{m+1,\Theta}(p)}(t^l_{j-1}) \\
& + \sum_{i,j} a_{i,j} \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} \mathbf{1}_{I^{(2)}_{[n/p+\Theta],\Theta}(p)}(t^k_{i-1}) \mathbf{1}_{\left[[n/p+\Theta]\frac{(p+\Theta)}{n}, 1\right)}(t^l_{j-1}) \\
& + \sum_{i,j} a_{i,j} \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} \mathbf{1}_{I^{(2)}_{[n/p+\Theta],\Theta}(p)}(t^l_{j-1}) \mathbf{1}_{\left[[n/p+\Theta]\frac{(p+\Theta)}{n}, 1\right)}(t^k_{i-1}) \\
& + \sum_{i,j} a_{i,j} \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} \mathbf{1}_{\left[[n/p+\Theta]\frac{(p+\Theta)}{n}, 1\right)}(t^l_{j-1}) \mathbf{1}_{\left[[n/p+\Theta]\frac{(p+\Theta)}{n}, 1\right)}(t^k_{i-1}),
\end{aligned}$$

for an arbitrary collection $(a_{i,j})$. Thus, in view that $Y^{(u),k,x,l,y}$ is again a semimartingale, we can apply Assumption 1, (67), Remark 2 and similar arguments as in the preceding sections to obtain that for $n$ and $p$ large

$$\begin{aligned}
U_n = \sqrt{n}\left(\psi^{kl}(H^{(u)}) - \int_0^1 H^{(u)}_s \Sigma_s ds\right) = & \sqrt{n} \sum_{m=1}^{[n/(p+\Theta)]} \xi^{(u),k,l}_m(p) \\
& + \sum_{r=1}^{5} \sqrt{n} \sum_{m=1}^{[n/(p+\Theta)]} \sum_{i,j} \zeta^{(u),k,l}_{i,j}(r,p) \mathbf{1}_{I^k_i \cap I^l_j \neq \emptyset} + \mathrm{o}_{\mathbb{P}}(1),
\end{aligned}$$

where

$$
\begin{aligned}
\xi_m^{(u),k,l}(p) &= \sum_{i,j}\sum_{x,y} Y^{(u),k,x,l,y}_{\min I^{(1)}_{m,\Theta}(p)} \tilde{\chi}^{kl}_{i,j}(x,y)\mathbf{1}_{I^{(1)}_{m,\Theta}(p)}(t^k_{i-1})\mathbf{1}_{I^{(1)}_{m,\Theta}(p)}(t^l_{j-1})\mathbf{1}_{I^k_i\cap I^l_j\neq\emptyset};\\
\zeta^{(u),k,l}_{i,j}(1,p) &= \sum_{x,y} Y^{(u),k,x,l,y}_{\min I^{(2)}_{m,\Theta}(p)} \tilde{\chi}^{kl}_{i,j}(x,y)\mathbf{1}_{I^{(2)}_{m,\Theta}(p)}(t^k_{i-1})\mathbf{1}_{I^{(2)}_{m,\Theta}(p)}(t^l_{j-1});\\
\zeta^{(u),k,l}_{i,j}(2,p) &= \sum_{x,y} Y^{(u),k,x,l,y}_{\min I^{(1)}_{m,\Theta}(p)} \tilde{\chi}^{kl}_{i,j}(x,y)\mathbf{1}_{I^{(1)}_{m,\Theta}(p)}(t^k_{i-1})\mathbf{1}_{I^{(2)}_{m,\Theta}(p)}(t^l_{j-1});\\
\zeta^{(u),k,l}_{i,j}(3,p) &= \sum_{x,y} Y^{(u),k,x,l,y}_{\min I^{(2)}_{m-1,\Theta}(p)} \tilde{\chi}^{kl}_{i,j}(x,y)\mathbf{1}_{I^{(1)}_{m,\Theta}(p)}(t^k_{i-1})\mathbf{1}_{I^{(2)}_{m-1,\Theta}(p)}(t^l_{j-1});\\
\zeta^{(u),k,l}_{i,j}(4,p) &= \sum_{x,y} Y^{(u),k,x,l,y}_{\min I^{(1)}_{m,\Theta}(p)} \tilde{\chi}^{kl}_{i,j}(x,y)\mathbf{1}_{I^{(2)}_{m,\Theta}(p)}(t^k_{i-1})\mathbf{1}_{I^{(1)}_{m,\Theta}(p)}(t^l_{j-1});\\
\zeta^{(u),k,l}_{i,j}(5,p) &= \sum_{x,y} Y^{(u),k,x,l,y}_{\min I^{(2)}_{m,\Theta}(p)} \tilde{\chi}^{kl}_{i,j}(x,y)\mathbf{1}_{I^{(2)}_{m,\Theta}(p)}(t^k_{i-1})\mathbf{1}_{I^{(1)}_{m+1,\Theta}(p)}(t^l_{j-1}),
\end{aligned}
$$

in which

$$
\tilde{\chi}^{kl}_{i,j}(x,y) = \Delta_{i,k}B^x\Delta_{j,l}B^y - Leb(I^k_i\cap I^k_i)\delta^{x,y}.
$$

Furthermore, by letting $V_n(p) = \sqrt{n}\sum_{m=1}^{[n/(p+\Theta)]}\sum_{i,j}\xi^{(u),k,l}_{i,j}(p)\mathbf{1}_{I^k_i\cap I^l_j\neq\emptyset}$ and following the argument in 6.1.5 in Christensen et al. (2013), we deduce that for all $\varepsilon>0$

$$
\lim_{p\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(|U_n - V_n(p)|>\varepsilon\right)=0.
$$

Hence, in view of the fact that $\xi^{(u),k,l}_m(p)$ is $\mathcal{G}^{p,n}_m = \mathcal{F}_{\frac{m(p+\Theta)}{n}}$ measurable, we deduce from Proposition 2.2.4 and Theorem 2.2.15 in Jacod and Protter (2011) that the conclusion of Theorem 2 is valid whenever for all $k,l,k',l'=1,\ldots,d$ and $u,v=1,\ldots,N$

$$
\begin{aligned}
&1.\ n^2\sum_{m=1}^{[n/(p+\Theta)]}\mathbb{E}\left(\left|\xi^{(u),k,l}_m(p)\right|^4\right)\xrightarrow[n\to\infty]{}0;\\
&2.\ \sqrt{n}\sum_{m=1}^{[n/(p+\Theta)]}\mathbb{E}\left(\xi^{(u),k,l}_m(p)\left(B^{k'}_{\max I_{m,\Theta}(p)}-B^{k'}_{\min I_{m,\Theta}(p)}\right)\Big|\,\mathcal{G}^{p,n}_{m-1}\right)\xrightarrow[n\to\infty]{\mathbb{P}}0;\\
&3.\ \sqrt{n}\sum_{m=1}^{[n/(p+\Theta)]}\mathbb{E}\left(\xi^{(u),k,l}_m(p)\left(N_{\max I_{m,\Theta}(p)}-N_{\min I_{m,\Theta}(p)}\right)\Big|\,\mathcal{G}^{p,n}_{m-1}\right)\xrightarrow[n\to\infty]{\mathbb{P}}0;\\
&4.\ n\sum_{m=1}^{[n/(p+\Theta)]}\mathbb{E}\left(\xi^{(u),k,l}_m(p)\xi^{(v),k',l'}_m(p)\Big|\,\mathcal{G}^{p,n}_{m-1}\right)\xrightarrow[n\to\infty]{\mathbb{P}}\Upsilon^{(u),k,l,(v),k',l'}_p;\\
&5.\ \Upsilon^{(u),k,l,(v),k',l'}_p\xrightarrow[p\to\infty]{\mathbb{P}}\int_0^1 H^{(u)}_sH^{(v)}_s\left(\gamma^{kk',ll'}\varSigma^{k,k'}_s\varSigma^{l,l'}_s+\gamma^{kl',l,k'}\varSigma^{k,l'}_s\varSigma^{k',l}_s\right)ds,
\end{aligned}
\tag{84}
$$

in which $N$ is a bounded martingale satisfying that $[N,B]=0$ and $\Upsilon_p^{(u),k,l,(v),k',l'}$ is a certain $\mathcal{F}$-measurable array.

In the remainder of this section we verify that 1.-3. hold, while conditions 4. and 5. are verified in the following subsection. A simple application of Jensen's inequality, Assumption 1 and (67) gives us that

$$\mathbb{E}\left(\left|\xi_m^{(u),k,l}(p)\right|^4\right)\leq C\sum_{i,j}\sum_{x,y}\mathbb{E}\left(\left|\tilde{\chi}_{i,j}^{kl}(x,y)\right|^4\right)\mathbf{1}_{I_{m,\Theta}^{(1)}(p)}(t_{i-1}^k)\mathbf{1}_{I_{m,\Theta}^{(1)}(p)}(t_{j-1}^l)\mathbf{1}_{I_i^k\cap I_j^l\neq\emptyset}\leq C/n^4,$$

which implies 1. Condition 2. follows easily from the independence of the increments of the Brownian motion. Since $[N,B]=0$, we infer from Itô's formula that for all $t_{i-1}^k,t_{l-1}^l\in I_{m,\Theta}(p)$ , such that $I_i^k\cap I_j^l\neq\emptyset$ it holds that

$$\mathbb{E}\left(\Delta_{i,k}B^x\Delta_{j,l}B^y\left(N_{\max I_{m,\Theta}(p)}-N_{\min I_{m,\Theta}(p)}\right)\Big|\,\mathcal{G}_{m-1}^{p,n}\right)=\mathbb{E}\left(\int_{I_i^k\cap I_j^l}dB_s^x\int_{I_i^k\cap I_j^l}dB_s^y\int_{I_i^k\cap I_j^l}dN_s\Big|\,\mathcal{G}_{m-1}^{p,n}\right)=0,$$

so 3. in (84) also holds.

### 11.4.1 Asymptotic variance

In this part we will show that 4. and 5. in (84) are satisfied. Using the fact that $B$ has independent increments, one easily deduces that

$$\mathbb{E}\left(\xi_m^{(u),k,l}(p)\xi_m^{(v),k',l'}(p)\Big|\,\mathcal{G}_{m-1}^{p,n}\right)=\tilde{Y}_{\min I_{m,\Theta}(p)}^{u,v,k,k',l,l'}\Gamma_m^{k,k',l,l'}+\tilde{Y}_{\min I_{m,\Theta}(p)}^{u,v,k,l',k',l}\tilde{\Gamma}_m^{k,l',k',l},$$

where $\tilde{Y}^{u,v,k,k',l,l'}=H_t^{(u)}H_t^{(v)}\Sigma_t^{k,k'}\Sigma_t^{l,l'}$ and

$$\Gamma_m^{k,k',l,l'}=\sum_{i,j}\sum_{i',j'}Leb(I_i^k\cap I_{i'}^{k'})Leb(I_j^l\cap I_{j'}^{l'})\mathbf{1}_{I_{m,\Theta}^{(1)}(p)}(t_{i-1}^k)\mathbf{1}_{I_{m,\Theta}^{(1)}(p)}(t_{j-1}^l)\mathbf{1}_{I_i^k\cap I_j^l\neq\emptyset}\mathbf{1}_{I_{m,\Theta}^{(1)}(p)}(t_{i'-1}^{k'})\mathbf{1}_{I_{m,\Theta}^{(1)}(p)}(t_{j'-1}^{l'})\mathbf{1}_{I_{i'}^{k'}\cap I_{j'}^{l'}\neq\emptyset};$$

$$\tilde{\Gamma}_m^{k,l',k',l}=\sum_{i,j}\sum_{i',j'}Leb(I_i^k\cap I_{j'}^{l'})Leb(I_j^l\cap I_{i'}^{k'})\mathbf{1}_{I_{m,\Theta}^{(1)}(p)}(t_{i-1}^k)\mathbf{1}_{I_{m,\Theta}^{(1)}(p)}(t_{j-1}^l)\mathbf{1}_{I_i^k\cap I_j^l\neq\emptyset}\mathbf{1}_{I_{m,\Theta}^{(1)}(p)}(t_{i'-1}^{k'})\mathbf{1}_{I_{m,\Theta}^{(1)}(p)}(t_{j'-1}^{l'})\mathbf{1}_{I_{i'}^{k'}\cap I_{j'}^{l'}\neq\emptyset}.$$

We proceed as in Christensen et al. (2013) and we concentrate on the interval

$$\tilde{I}_m(p)=\left[\frac{(m-1)(p+\Theta)+2\Theta}{n},\frac{(m-1)(p+\Theta)+p-2\Theta}{n}\right)$$

for $p>4\theta$. Fix now $t_{i-1}^k\in\tilde{I}_m(p)\subseteq I_{m,\Theta}^{(1)}(p)$. Note that $I_i^k\cap I_j^l\neq\emptyset$ if and only if

$$t_i^k>t_{j-1}^l\geq t_{i-1}^k,\ \text{or}\ t_{i-1}^k>t_{j-1}^l\geq t_{i-1}^k-\frac{1}{n_l};$$

or, in other words, for

$$\frac{n_l}{n_k}(i-1)\leq j<\frac{n_l}{n_k}i+1. \tag{85}$$

Similarly, $Leb(I_i^k \cap I_{i'}^{k'}) \neq 0$ and $Leb(I_j^l \cap I_{j'}^{l'})$ if and only if

$$\frac{n_{k'}}{n_k}(i-1) < i' < \frac{n_{k'}}{n_k}i + 1 \tag{86}$$

and

$$\frac{n_{l'}}{n_l}(j-1) < j' < \frac{n_{l'}}{n_l}j + 1, \tag{87}$$

respectively. Moreover, in view that

$$\frac{(m-1)(p+\Theta)+\Theta}{n} \leq t_{i-1}^k - \frac{1}{n_r} \text{ and } t_i^k < \frac{(m-1)(p+\Theta)+p-\Theta}{n}$$

for every $r = 1, \ldots, d$, then those times satisfying (85), (86) and (87) necessarily belong to $I_{m,\Theta}^{(1)}(p)$. Thus,

$$\begin{aligned}\Gamma_m^{k,k',l,l'} = & \sum_{t_{i-1}^k \in \tilde{I}_m(p)} \sum_{\frac{n_l}{n_k}(i-1) \leq j < \frac{n_l}{n_k}i+1} \sum_{\frac{n_{k'}}{n_k}(i-1) < i' < \frac{n_{k'}}{n_k}i+1} \sum_{\frac{n_{l'}}{n_l}(j-1) < j' < \frac{n_{l'}}{n_l}j+1} Leb(I_i^k \cap I_{i'}^{k'}) Leb(I_j^l \cap I_{j'}^{l'}) \mathbf{1}_{I_{i'}^{k'} \cap I_{j'}^{l'} \neq \emptyset} \\ & + \mathrm{O}(n^2) \\ =: & \bar{\Gamma}_m^{k,k',l,l'}(p) + \mathrm{O}(n^2).\end{aligned}$$

Furthermore,

$$n \sum_{m=1}^{[n/(p+\Theta)]} \tilde{Y}_{\min I_{m,\Theta}(p)}^{u,v,k,k',l,l'} \Gamma_m^{k,k',l,l'} = \frac{1}{n} \sum_{m=1}^{[n/(p+\Theta)]} \tilde{Y}_{\min I_{m,\Theta}(p)}^{u,v,k,k',l,l'} n^2 \bar{\Gamma}_m^{k,k',l,l'}(p) + \frac{1}{p+\Theta} \mathrm{O}(1).$$

From here we easily identify that

$$n^2 \bar{\Gamma}_m^{k,k',l,l'}(p) \to (p-4\Theta)\gamma^{k,k',l,l'}.$$

Consequently,

$$n \sum_{m=1}^{[n/(p+\Theta)]} \tilde{Y}_{\min I_{m,\Theta}(p)}^{u,v,k,k',l,l'} \Gamma_m^{k,k',l,l'} \xrightarrow{\mathbb{P}} \frac{(p-4\Theta)}{p+\Theta} \int_0^1 H_s^{(u)} H_s^{(v)} \gamma^{kk',ll'} \Sigma_s^{k,k'} \Sigma_s^{l,l'} ds + \mathrm{o}_{\mathbb{P}}(1).$$

In a similar way, it is possible to deduce that

$$n \sum_{m=1}^{[n/(p+\Theta)]} \tilde{Y}_{\min I_{m,\Theta}(p)}^{u,v,k,k',l,l'} \tilde{\Gamma}_m^{kl',k',l} \xrightarrow{\mathbb{P}} \frac{(p-4\Theta)}{p+\Theta} \int_0^1 H_s^{(u)} H_s^{(v)} \gamma^{kl',l,k'} \Sigma_s^{kl'} \Sigma_s^{k'l} ds + \mathrm{o}_{\mathbb{P}}(1).$$

Relations 4. and 5. in (84) are now obtained by letting $p \to \infty$.

# 12 Appendix 3: Consistency of the asymptotic covariance estimator

*Proof.* Itô's formula and (67) show that for all $t \geq 0$

$$\bar{\Sigma}_t^{kl} = \int_0^{T^{kl}(t)} \Sigma_s^{kl} ds + \sum_{i,j} \xi_{i,j} \mathbf{1}_{(0,t]}(t_i^k \vee t_j^l) \mathbf{1}_{I_i^k \cap I_j^l \neq \emptyset},$$

where $T^{kl}(t) = \frac{[n_k t]}{n_k} \wedge \frac{[n_l t]}{n_l}$ and for certain r.vs satisfying that

$$\left| \mathbb{E}\left( \xi_{i,j} | \mathcal{F}_{t_{j-1}^l \wedge t_{i-1}^k} \right) \right| \leq C/n^{3/2}, \;\; \mathbb{E}\left[ (\xi_{i,j})^2 \right] \leq C/n^2.$$

Thus, by arguments similar to those used in the preceding sections, we get that

$$\begin{aligned} \bar{\Sigma}_{t+k_n/n}^{kl} - \bar{\Sigma}_t^{kl} &= \Delta_{k_n}^n T^{kl}(t) \int_0^1 \Sigma_{T^{kl}(t)+\Delta_{k_n}^n T^{kl}(t) r}^{kl} dr + \sum_{i,j} \xi_{i,j} \mathbf{1}_{(t, t+\frac{k_n}{n}]}(t_i^k \vee t_j^l) \mathbf{1}_{I_i^k \cap I_j^l \neq \emptyset} \\ &= \Delta_{k_n}^n T^{kl}(t) \int_0^1 \Sigma_{T^{kl}(t)+\Delta_{k_n}^n T^{kl}(t) r}^{kl} dr + \mathrm{o}_{\mathbb{P}}(k_n/n), \end{aligned}$$

in which we have let $\Delta_{k_n}^n T^{kl}(t) = T^{kl}(t+k_n/n) - T^{kl}(t+k_n/n)$. Furthermore, using that $x - [x] \leq 1$, we conclude that

$$\frac{n}{k_n} \Delta_{k_n}^n T^{kl}(t) \to 1.$$

This implies that

$$\tilde{\Sigma}_t^{k',l'} = \Sigma_t^{kl} + \mathrm{o}_{\mathbb{P}}(1).$$

Replacing $n$ by $n_k$ results in a similar statement for $\hat{\Sigma}_{t_{m-1}^k}^{kl}$. We conclude that

$$\hat{\Psi}^{k',l'} = C_\theta \int_0^1 f_n(s) ds + \mathrm{o}_{\mathbb{P}}(1),$$

where

$$f_n(s) = \sum_{kl}^{d} \sum_{m,p} \left( \tilde{\Sigma}_{t_{m-1}^k}^{k',l'} w_{t_{m-1}^k}^k \hat{\Sigma}_{t_{m-1}^k}^{kl} w_{t_{p-1}^l}^l + \hat{\Sigma}_{t_{m-1}^k}^{k,k'} w_{t_{m-1}^k}^k \hat{\Sigma}_{t_{p-1}^l}^{l,l'} w_{t_{p-1}^l}^l \right) \mathbf{1}_{I_m^k}(s) \mathbf{1}_{I_p^l}(s), \;\; 0 \leq s \leq 1,$$

which converges for almost all $s$ toward

$$\Sigma_s^{k',l'} w_s^k \Sigma_s^{kl} w_s^j ds + \Sigma_s^{k,k'} w_s^k \Sigma_s^{l,l'} w_s^l.$$

The Dominated Convergence Theorem and Remark 2 conclude the proof.

□

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