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WORKING PAPER

Taylor rules and the Canadian-US equilibrium exchange rate

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February 2010

2010/643

D/2010/7012/14

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February 4, 2010

Abstract

This paper identifies the Canadian-US equilibrium exchange rate based on a simple structural model of the real exchange rate, in which monetary policy follows a Taylor rule interest rate reaction function. The equilibrium exchange rate is explained by relative output and inflation as observable variables, and by unobserved equilibrium rates as well as unobserved transitory components in output and the exchange rate. Using Canadian data over 1974-2008 we jointly estimate the unobserved components and the structural parameters using the Kalman filter and Bayesian technique. We find that Canada's equilibrium exchange rate evolves smoothly and follows a trend depreciation. The transitory component is found to be very persistent but much more volatile than the equilibrium rate, resulting in few but prolonged periods of currency misalignments.

Keywords: equilibrium exchange rate, unobserved components, Kalman filter, Bayesian analysis, Importance sampling

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1 Introduction

The identification and estimation of equilibrium exchange rates is a controversial topic in international macroeconomics. The literature has come up with a number of different ways of determining equilibrium rates, and results strongly depend on which particular approach is used. Yet knowledge of equilibrium rates is indispensable for a variety of issues in exchange rate economics, including assessments of currency misalignments, the decision of opting for fixed or flexible exchange rates, or questions regarding the reform of the international monetary system. It is also of particular relevance when large movements in the exchange rate coincide with broad stability in economic fundamentals, as was recently experienced in Canada (OECD, 2004, p. 53).

In this paper we propose a new approach to estimating equilibrium exchange rates. Our approach is based on a simple structural two-country open economy model, in which monetary policy is described by Taylor rule reaction functions. Standard monetary models of exchange rate determination have long been discredited by their failure to explain exchange rate behavior, as forcefully documented by Meese and Rogoff (1983), Meese (1990), or Flood and Rose (1995). A new strand of literature allows for the endogeneity of monetary policy by incorporating Taylor rule reaction functions into otherwise standard exchange rate models (Engel and West, 2004, 2005; Bacchetta and van Wincoop, 2006). Such models display exchange rate behavior quite different from traditional exchange rate models. For example, whereas in standard flexible-price monetary models an increase in the current inflation rate causes the exchange rate to depreciate, in Taylor rule models the exchange rate appreciates because higher inflation induces expectations of tighter future monetary policy (Clarida and Waldman, 2007).

The emerging evidence on the empirical performance of Taylor rule models of the open economy is quite encouraging. Engel and West (2006) and Mark (2009) use the forecasts from VAR models for the fundamentals as measures of exchange rate expectations and compare the properties of exchange rates generated from such models with actual German-US bilaterals. Both are shown to be highly volatile and persistent, and the Taylor rule exchange rate turns out to be substantially more strongly correlated with the actual data than exchange rates generated by traditional models. Molodtsova and Papell (2009) analyze the out-of-sample predictability of exchange rates with Taylor rule fundamentals by employing an error-correction formulation for the Taylor rule model. They find that the evidence of exchange rate predictability is much stronger compared to conventional models, particularly at shorter forecast horizons.

We base our analysis on a variant of the two-country Taylor rule model introduced by Engel and

West (2006). In the model, the two countries differ in terms of the macroeconomic fundamentals included in their respective monetary reaction functions. Beside (expected) inflation and the output gap, the Taylor rule of one of the two countries also contains the real exchange rate as an argument. This feature is frequently, although not exclusively, associated with the small country assumption.[1] In this paper, we utilize the model to identify the Canadian-US equilibrium exchange rate. Canada is an archetypical small open economy, and the Bank of Canada has traditionally engaged in exchange market management, with the bilateral Canadian-US exchange rate as the primary target of these intervention activities (Weymark, 1995, 1997).

Whereas Engel and West (2006) use their model to explain the real exchange rate exclusively in terms of observable macroeconomic aggregates, we link these fundamentals with the transitory component of the exchange rate only, and let both the transitory and the long-run equilibrium real exchange rates be also influenced by random determinants. The unobserved components and the structural parameters are jointly estimated in a Bayesian framework.

The remainder of the paper is structured as follows: Section 2 provides a brief overview of the various concepts of equilibrium real exchange rates, Section 3 introduces the stylized small open economy model, Section 4 elaborates on our estimation methodology, Section 5 presents the estimation results, and Section 6 concludes.

2 Equilibrium real exchange rates

Equilibrium real exchange rates can be identified in various different ways. The most commonly used are the (enhanced) purchasing power parity (PPP), the fundamental equilibrium exchange rate (FEER), the behavioral equilibrium exchange rate (BEER), and the permanent equilibrium exchange rate (PEER).[2]

The simplest approach to determining equilibrium exchange rates is based on PPP, according to which an exchange rate is in equilibrium if it equalizes the purchasing power of national currencies in terms of particular goods or output bundles. A variant of this paradigm is the so-called enhanced PPP approach, which incorporates the Balassa-Samuelson effect by linking nations' per-capita income levels with their effective real exchange rates. Consequently, equilibrium real exchange rates should be weaker for low-income and emerging economies in comparison to (tech-

[1] Based on evidence that the real exchange rate enters an interest rate rule for Germany with a small, but statistically significant coefficient, Engel and West (2006) apply the model to the German-US real exchange rate.

[2] For more complete taxonomies of equilibrium exchange rates, see MacDonald (2000) and Driver and Westaway (2004).

nologically) more advanced countries. The empirical observation that exchange rates converge to their PPP equilibrium levels far too slowly to be compatible with any sensible notion of goods market arbitrage, the so-called PPP puzzle (Rogoff, 1996), implies that this equilibrium concept may determine the equilibrium exchange rate in the very long run only. In particular, this concept leaves out all factors which may account for deviations from PPP levels in terms of a time-varying equilibrium path of the exchange rate. These factors may include aggregate activity levels, net asset levels, or balance of payments positions, and are incorporated in various ways in the FEER and BEER equilibrium concepts.

FEERs have been popularized by Williamson (1983) as a concept of macroeconomic balance. This approach considers a country's internal and external balance, where the internal equilibrium corresponds to a zero output gap consistent with the NAIRU, and the external balance requires a sustainable current account position. This concept has been widely used by the IMF as the basis for the first and third approaches to estimating equilibrium exchange rates (IMF, 2006). However, the notion of a sustainable current account is not immediately operational. There is substantial uncertainty as to the exact magnitude of a "sustainable" external balance and whether divergences of the current account balance from target are transitory or permanent.

The natural real exchange rate (NATREX) has been introduced by Stein (1994) as an extension of the FEER based on dynamic stock-flow models, in which the external balance is explicitly modeled in terms of the key determinants of national savings and investment levels. These include the rate of time preference and the stock of foreign assets in the savings function, and the level and productivity of the capital stock in the investment function. Although appealing as a theoretical concept, empirical implementations of the NATREX have to rely on proxies for the most crucial variables in terms of the rate of time preference and the productivity of capital. For example, Stein uses the ratio of the sum of private and public consumption to GNP as the time preference measure, and a moving average of the growth of real GDP as the measure of productivity.

BEERs attempt to econometrically model the behavior of real exchange rates. Pioneered by Clark and MacDonald (1999), this approach tries to connect the observed real exchange rate with its long-run fundamental determinants, such as the terms of trade, the relative price of traded to non-traded goods, and net foreign assets. The relationship between the unobserved equilibrium exchange rate and the fundamentals is then assumed to be identical to the empirically estimated long-term relationship. BEERs are also used as the basis for the second of the IMF calculations of equilibrium exchange rates (IMF, 2006). A major drawback of BEERs lies in the assumption that the exchange rate is on average in equilibrium over the estimation period. Hence BEERs can only be used as an indicator of a country's under- or overvaluation relative to its own past averages and not as an absolute measure of the equilibrium exchange rate (Cline and Williamson, 2007).

A general problem of both the FEER and BEER approaches regards the selection and measurement of the appropriate fundamentals. There is a wide array of potentially important variables, and the outcomes depend critically on the set of variables included in the set of relevant fundamentals. Apart from selection issues, it is far from obvious how to measure the long-term values of the fundamentals themselves. Possibly most worrying is the observation that the influence of the variables most frequently included in the set of fundamentals of equilibrium exchange rates, such as the terms of trade or the stock of net foreign assets, is empirically not substantiated (Egert et al., 2006).

As an alternative approach which avoids problems associated with the selection of fundamental variables, PEERs use time-series estimators to decompose real exchange rates into their permanent and transitory components, with the permanent component defined as a measure of the equilibrium exchange rate. Such decompositions can be obtained by means of various statistical techniques, such as univariate or multivariate Beveridge-Nelson decompositions, structural vector-autoregressions, or cointegration-based estimation techniques.[3] A major disadvantage of PEERs lies in the fact that such decompositions are purely statistical and incorporate no economic determinants of exchange rate equilibrium.

In addressing the lack of economic determinants in PEER decompositions of the real exchange rate, a number of papers combine the BEER approach with a PEER decomposition, and use the latter for assessment purposes, (see e.g. Alberola et al., 1999; Hoffmann and MacDonald, 2001; Clark and MacDonald, 2004). The approach in this paper also combines the BEER and PEER approaches. However, our analysis differs from these earlier studies in two important respects. First, we derive the set of fundamentals from a well-specified open-economy model. And second, we let these fundamentals affect the exchange rate through its transitory component only, whereas both the transitory component and the long-run equilibrium real exchange rate are also driven by random determinants. We associate the latter with a set of (unspecified) fundamentals, such as those identified in the various BEER approaches.

[3]See MacDonald (2000) for an overview and further references.

3 A stylized two-country model with Taylor rules

We follow Engel and West (2004) in specifying a two-country model of the open economy, in which monetary policy in the home and foreign economies are described by Taylor rules, respectively given by

$$i_t - \bar{i}_t = \gamma_q \tilde{q}_t + \gamma_\pi E_t(\pi_{t+1}) + \gamma_y \tilde{y}_t + u_{mt}, \tag{1}$$

and

$$i_t^* - \bar{i}_t = \gamma_\pi E_t(\pi_{t+1}^*) + \gamma_y \tilde{y}_t^* + u_{mt}^*, \tag{2}$$

where asterisks denote foreign variables. In equations (1) and (2), i_t and i_t^* are the home and foreign interest rates in time period t, with $\bar{i}_t$ the corresponding natural interest rate, assumed to be identical in both countries. $E_t(\pi_{t+1})$ and $E_t(\pi_{t+1}^*)$ denote inflation expectations relative to target at home and abroad. $\tilde{y}_t$ and $\tilde{y}_t^*$ represent the domestic and foreign output gaps, and u_{mt} and u_{mt}^* are shocks to the home and foreign monetary policy rules. Assuming all coefficients of the policy rules to be positive and identical at home and abroad, the only difference arises with the inclusion of the term $\tilde{q}_t$ in the Taylor rule of the home country. Define the (log) real exchange rate as $q_t = e_t - p_t - p_t^*$, with e_t the (log) nominal exchange rate, expressed as the home currency price of foreign exchange, p_t and p_t^* the (log) domestic and foreign price levels, and $\tilde{q}_t$ as the deviation of the real exchange rate from its equilibrium level. A positive coefficient γ_q implies that the domestic monetary authority is assumed to raise the interest rate whenever the real exchange rate is undervalued relative to its equilibrium level. The Taylor rule of the domestic economy can thus alternatively be viewed as a monetary conditions index (MCI).[4]

Exchange rate expectations enter the model via the uncovered interest rate parity condition

$$i_t - i_t^* = E_t(e_{t+1}) - e_t. \tag{3}$$

Using the definition of the real exchange rate, (3) can be rewritten as

$$i_t - i_t^* = E_t[\pi_{t+1} - \pi_{t+1}^*] + E_t(q_{t+1}) - q_t. \tag{4}$$

[4] MCIs have been analyzed extensively in the recent literature both as a theoretical concept (e.g. Ball, 1999; Svensson, 2000; Batini et al., 2003) and as an empirical approximation to the actual conduct of monetary policy, particularly for small open economies (e.g. Freedman, 1994; Clarida et al., 1998; Gerlach and Smets, 2000).

The observed real exchange rate, q_t, can be expressed as the sum of its equilibrium rate, $\bar{q}_t$, which we specify as a random walk, and a transitory component, $\tilde{q}_t$, defined as above, such that

$$q_t = \bar{q}_t + \tilde{q}_t, \tag{5}$$

and

$$\bar{q}_t = \bar{q}_{t-1} + \eta_{1t}, \tag{6}$$

with η_{1t} a Gaussian mean zero white noise error term. Substituting (1), (2) and (5) into (4), and noting that $E_t(\bar{q}_{t+1}) = \bar{q}_t$, yields an expression for the transitory component of the real exchange rate in terms of a set of Taylor rule fundamentals, given by

$$\tilde{q}_t = \phi E_t(\tilde{q}_{t+1}) + \phi(1-\gamma_\pi)E_t[\pi_{t+1} - \pi^*_{t+1}] - \phi\gamma_y[\tilde{y}_t - \tilde{y}^*_t] + \eta_{2t}, \tag{7}$$

where $\phi = 1/(1+\gamma_q)$ and $\eta_{2t} = -b(u_{mt} - u^*_{mt})$. Equation (7) expresses the transitory exchange rate as a weighted average of inflation and output gap differentials between the home and foreign economies. The error term of the equilibrium exchange rate, η_{1t}, can thus be interpreted as comprising the set of all relevant exchange rate fundamentals other than those already captured in the specification of the transitory real exchange rate. Equation (7) includes expected values of the transitory exchange rate and the rates of inflation. As these variables are not observable we proxy them by their respective lagged values. This way of modeling expectations is frequently encountered in the literature, particularly for inflation (see e.g. Basistha, 2007), and can be motivated by the assumption of adaptive expectations, which is particularly adequate if the variable under consideration exhibits substantial persistence [5]. This results in the following estimable expression

$$\tilde{q}_t = \phi(L)\tilde{q}_{t-1} + \phi(L)(1-\gamma_\pi)[\pi_{t-1} - \pi^*_{t-1}] - \phi\gamma_y[\tilde{y}_t - \tilde{y}^*_t] + \eta_{2t}, \tag{7'}$$

where the lag polynomial is defined as $\phi(L) = \phi_1 + \phi_2 L + ... + \phi_q L^q$. Equation (7′) is akin to the (equilibrium) exchange rate equation of traditional monetary exchange rate models (see Frankel, 1979, p. 612). The major difference arises with respect to the coefficient on the inflation

[5] Inflation persistence has been ascertained not only for the US and Canada, but also for a number of other of industrialized countries. For an early authoritative study, see Gerlach and Smets (1997).

differential, which is positive in traditional models, but turns out to be negative in Taylor rule models as long as the Taylor principle ($\gamma_\pi > 1$) is satisfied. In monetary models, any increase in home relative to foreign inflation causes an excess supply of home relative to foreign money, thus depreciating the home currency. However, if central banks adhere to the Taylor principle, an increase in the home relative inflation rate is met by a rising real interest rate differential, thus inducing a home currency appreciation.

While we allow the transitory exchange rate to be influenced by economic fundamentals, the equilibrium exchange rate is assumed to evolve according to a unit-root process. The unit-root specification of $\bar{q}_t$ implies that the actual exchange rate is non-stationary.[6] One way of accounting for the non-stationarity of real exchange rates is through Balassa-Samuelson effects. According to the Balassa-Samuelson hypothesis (Balassa, 1964; Samuelson, 1964), technological change should have a bigger impact on tradable relative to non-tradable goods, since the latter are primarily labor-intensive services. Through its impact on wages, a lower trend rate of technological progress in Canada as compared to the US should therefore induce the Canada-US real exchange rate to depreciate over time.[7] We specifically allow for this effect by including the difference of potential per-capita output between Canada and the US as an explanatory variable in the equilibrium real exchange rate equation. Using the change in potential per-capita output as a proxy for the rate of technological change has the advantage of stripping out the influence of both differential population growth rates and differences in the business-cycle positions between Canada and the US on our measure of technology.

Accordingly, we adjust equation (6) to include a potential Balassa-Samuelson effect. The equilibrium exchange rate is then given by

$$\bar{q}_t = \bar{q}_{t-1} + \beta(\bar{y}_{t-1} - \bar{y}^*_{t-1}) + \eta_{1t}, \tag{6'}$$

where $\bar{y}_t$ and $\bar{y}^*_t$ now denote potential per-capita output in the home and foreign economies.[8]

The model is closed by specifying equations for aggregate output in the home and the foreign economies. As with the real exchange rate, current output can be decomposed into permanent

[6] Although purchasing power parity (PPP) and the stability of real exchange rates continue to be the subject of much academic debate, the evidence in favor of PPP usually comes out strong only in long data sets (see the collection of papers in Taylor, 2009). By applying standard unit-root tests we cannot reject the null hypothesis of a unit-root in the exchange rate analyzed here. Detailed results are available on request.

[7] For example, Clark and MacDonald (2004) find the secular downward trend in the relative price of Canadian non-traded to traded goods during the second half of the 20th century as being driven in part by a Balassa-Samuelson effect.

[8] For purposes of consistency, we use per-capita rather than aggregate measures for all permanent and transitory output components in the empirical analysis below.

and transitory components. The permanent component, often referred to as potential output, is modeled as a random walk with drift and is denoted by $\bar{y}_t$. The transitory component, denoted by $\tilde{y}_t$, is simply the output gap as defined above, which is modeled as a stationary autoregressive process. Hence, aggregate output in the home country is given by

$$y_t = \tilde{y}_t + \bar{y}_t \tag{8}$$

$$\tilde{y}_t = \kappa(L)\tilde{y}_{t-1} + \eta_{3t} \tag{9}$$

$$\bar{y}_t = \bar{y}_{t-1} + \mu_h + \eta_{4t}. \tag{10}$$

and output in the foreign country is given by

$$y_t^* = \tilde{y}_t^* + \bar{y}_t^* \tag{11}$$

$$\tilde{y}_t^* = \delta(L)\tilde{y}_{t-1}^* + \eta_{5t} \tag{12}$$

$$\bar{y}_t^* = \bar{y}_{t-1}^* + \mu_f + \eta_{6t}, \tag{13}$$

The model presented here can be seen as an extension of the approach presented in Engel and Kim (1999). Similar to their analysis we decompose the exchange rate into transitory and permanent components and estimate the model using the Kalman filter and Bayesian technique. However, our multivariate model relates the transitory component to the output gap and the rate of inflation via the home and foreign Taylor rules. Thus, it uses information contained in these variables to better identify the transitory component and consequently the equilibrium rate. Multivariate decompositions of macroeconomic variables have been shown to considerably reduce the uncertainty regarding the estimation of the unobserved variables. [9]

[9] For example, Basistha and Startz (2008) show that a multivariate unobserved component model to estimate the natural rate of unemployment cuts in half uncertainty as measured by variance and leads to significantly tighter confidence bands as compared to a univariate decomposition.

4 Estimation methodology

4.1 State space representation of the model

The model given by equations (5)-(13) can be cast into a linear Gaussian state space model of the following general form[10]

$$y_t = Z\alpha_t + Ax_t + \varepsilon_t, \qquad \varepsilon_t \sim N(0, H), \tag{14}$$

$$\alpha_{t+1} = S_t + T\alpha_t + \eta_t, \qquad \eta_t \sim N(0, Q), \qquad t = 1, \ldots, n, \tag{15}$$

where y_t is a $p \times 1$ vector of p observed endogenous variables, modeled in the observation equation (14), x_t is a $k \times 1$ vector of k observed exogenous or predetermined variables and α_t is a $m \times 1$ vector of m unobserved states, modeled in the state equation (15). The vectors ε_t and η_t are assumed to hold mutually independent Gaussian error terms with the former representing measurement errors and the latter structural shocks. The exact specification of the vectors y_t, x_t and α_t and the matrices Z, S, A, T, R, H and Q is provided in Appendix A.1.

4.2 Parameter estimation: a Bayesian framework

For given parameter matrices Z, A, T, S, H, and Q, the unobserved state vector α_t can be identified from the observations $y_1, \ldots, y_n$ and $x_1, \ldots, x_n$ using the Kalman filter and smoother. In practice these matrices generally depend on elements of an unknown parameter vector ψ. One possible approach is to derive the log-likelihood function for the model under study from the Kalman filter (see e.g. de Jong, 1991; Koopman and Durbin, 2000; Durbin and Koopman, 2001) and replace the unknown parameter vector ψ by its maximum likelihood (ML) estimate. This is not the approach pursued in this paper. We analyze the state space model from a Bayesian point of view, i.e. we use prior information to down-weight the likelihood function in regions of the parameter space that are inconsistent with out-of-sample information and/or in which the structural model is not interpretable (Schorfheide, 2006). More formally, we treat ψ as a random parameter vector with a known prior density $p(\psi)$ and estimate the posterior densities $p(\psi \mid y, x)$ for the parameter vector ψ and $p(\widehat{\alpha}_t \mid y, x)$ for the smoothed state vector $\widehat{\alpha}_t$, where y and x denote the stacked vectors $(y_1', \ldots, y_n')'$ and $(x_1', \ldots, x_n')'$ respectively, by combining information

[10] See e.g. Durbin and Koopman (2001) for an extensive overview of state space models.

contained in $p(\psi)$ and the sample data. This boils down to calculating the posterior mean $\overline{g}$

$$\overline{g} = E\left[g(\psi) \mid y, x\right] = \int g(\psi)\, p(\psi \mid y, x)\, d\psi, \tag{16}$$

where g is a function which expresses the moments of the posterior densities $p(\psi \mid y, x)$ and $p(\widehat{\alpha}_t \mid y, x)$ in terms of the parameter vector ψ. In principle, the integral in equation (16) can be evaluated numerically by drawing a sample of n random draws of ψ, denoted $\psi^{(i)}$ with $i = 1, \ldots, n$, from $p(\psi \mid y, x)$ and then estimating $\overline{g}$ by the sample mean of $g(\psi)$. As $p(\psi \mid y, x)$ is not a density with known analytical properties, such a direct sampling method is not feasible, though. Therefore, we use importance sampling (see Appendix A.2 for technical details).

As noted by Planas et al. (2008), Bayesian estimation of unobserved component models avoids the pile-up problem by specifying prior distributions that are strictly positive for the variance parameters. Another important advantage of the Bayesian framework over standard ML is that it is straightforward to calculate the posterior densities of both the parameter vector ψ and the smoothed state vector $\widehat{\alpha}_t$ where the latter takes both parameter and filter uncertainty into account (see Appendix A.3 for technical details).

5 Estimation Results[11]

5.1 Data

We use quarterly data for Canada and the US from 1974Q1 to 2009Q1 taken from the International Monetary Fund (IMF) International Financial Statistics. For inflation we use the first difference of the log of the seasonally adjusted CPI. Per-capita output is seasonally adjusted quarterly GDP divided by total population. Output per capita and the real exchange rate are in natural logs multiplied by 100. The transitory components in output and the exchange rates can thus be interpreted as percentage deviations form their long-run values. Starting in 1974 implies (i) that we only focus on the post Bretton Woods era and (ii) that we do not need to address the productivity slowdown in real GDP in the early 1970s.

[11] The GAUSS code to obtain the results presented in this section is available on request.

5.2 Structural breaks in the mean of inflation

Relative inflation is an exogenous variable in our model. To construct this variable we have to demean the country specific inflation rates. Various studies have shown that there are mean breaks in the rate of inflation in Canada and the US (see e.g. Basistha (2007) for Canada and Rapach and Wohar (2005) for US data). Thus we demean the inflation rates but allow for frequent changes in the mean. The number and the timing of the breaks is determined by the (Bai and Perron, 1998, 2001, 2003, hereafter BP) structural break test.[12]

Table 1 presents the results of the BP tests on structural breaks in inflation.[13]

Table 1 about here

Both the WD_{max} and the UD_{max} test statistic clearly reject the null hypothesis of no structural breaks in Canadian inflation at conventional confidence levels. The sequential analysis also rejects the null hypothesis of no breaks against the alternative hypothesis of one break as well as the null of one break against the alternative hypothesis of two structural breaks. However, more than two breaks are not found. The detected break dates are 1982:Q3 and 1991:Q1.[14] For the US we find one break in 1982:Q4.

5.3 Prior distribution of the parameters

Prior information on the unknown parameter vector ψ is included in the analysis through the prior density $p(\psi)$. Detailed information on $p(\psi)$ can be found in the first columns of Table 2. As stated above, the main motivation for setting these priors is to down-weight the likelihood function in regions of the parameter space that are inconsistent with out-of-sample information and/or in which the structural model is not interpretable. Previous estimates as well as economic theory give us an idea about the approximate value of the model's parameters. However, using previous studies to set priors should be done with caution particularly if theses studies consider the same time period. We therefore use previous estimates only as a rough indication for the prior means but choose the prior variance fairly loose. The bivariate unobserved component model for Canada

[12]Briefly, BP suggest to first examine two tests (the so called UD_{max} and WD_{max} tests) to check if there are any structural breaks. If these tests reject the null of no breaks, a sequential procedure to determine the number of breaks is used. According to the BP notation this means computing a sequence of $\text{Sup}F_T(l+1|l)$ statistics to test the null of l breaks against the alternative of $l+1$ breaks. A detailed description of this test can be found in BP and Rapach and Wohar (2005).

[13]The results of the BP tests have been obtained by using the original GAUSS program from P. Perron available on his webpage.

[14]The break dates are similar to the one in Basistha (2007).

of Basistha (2007) provides an indication for the parameter values in output. There are several studies estimating potential output and the output gap for the US economy (see e.g Basistha and Nelson, 2007)

γ_π and γ_y play a crucial rule in determining the impact of relative inflation and the relative output gap on transitory exchange rate movements. In a Taylor rule these parameters measure the relative weight given to inflation and the output gap given by the central bank when setting the interest rate. We follow the literature and set $\gamma_\pi = 1.5$ and $\gamma_y = 0.5$ (see e.g. Engel and West, 2006). The variance parameters of the transitory and the permanent component of the exchange rate determine the variability of these components and therefore are crucial for the decomposition. We use non-informative priors on $\sigma^2_{\eta_1}$ and $\sigma^2_{\eta_2}$ in order to 'let the data speak'. Regarding a potential Balassa-Samuelson effect we also use a very loose prior on β with mean zero.

All transitory components include two lagged dependent variables.[15] The prior distribution of the autoregressive parameters is chosen so that its 90% interval covers the range $[0.14, 0.86]$. Thus the prior distribution does neither impose very volatile nor very persistent transitory components. For potential output growth we set the prior to 0.45. The prior variance is chosen such that the 90% interval for the annualized growth rate of potential output per capita in both countries ranges from 1.28% to 2.32%.

5.4 Posterior distribution of the parameters

The last two columns of Table 2 show the posterior mean and the 10% and 90% percentiles of the posterior distribution of all parameters. Similar to other studies for industrialized countries we find the output gaps to be relatively persistent processes. Similarly, the sum of the AR parameters in the exchange rate gap is 0.96, implying a high degree of persistence. However, the result of very persistent transitory shocks is consistent with other estimates in the literature (see e.g Engel and Kim, 1999; Rogoff, 1996). By setting the prior mean for γ_π to 1.5 we implicitly assume that relative inflation and the transitory exchange rate are negatively related. The posterior mean of γ_π is 1.17 with a 90% interval ranging from 1.07 to 1.27. Although the posterior mean is outside the prior 90% interval, it is above unity, thereby confirming the negative relation between the relative inflation and the transitory exchange rate.[16] Figure 4 shows the prior together with the

[15]The AR(2) specification is standard for output gap estimates when quarterly data are used. We experimented with a different lag length for the transitory component of the exchange rate but found that an AR(2) process fits the data best.

[16]The same qualitative result obtains even with a diffuse prior variance, resulting in an inflation coefficient of γ_π=1.08.

posterior distribution for all parameters.

Table 2 about here

5.5 Posterior distribution of the states

Figure 1 shows the smoothed estimates of the transitory exchange rate and the equilibrium exchange rate together with actual output. The equilibrium rate evolves smoothly as compared to the transitory component. It turns out that the equilibrium exchange rate is far from being constant as it exhibits a trend depreciation over the sample period. Thus a first result is that simple demeaning of the bilateral Canadian-US real exchange rate leads to an incorrect measure of deviations from its equilibrium level. The transitory component is found to often deviate substantially from the equilibrium rate. Moreover these deviations are very persistent.

Figure 1 about here

Figures 2 and 3 display the smoothed estimates of the output gaps and potential output for Canada and the US. The shaded areas indicate recessions as defined by the Economic Cycle Research Institute for Canada and the NBER for the US economy. The estimated output gap picks up the business cycle turning points quite accurately. Shape and magnitude of the output gaps correspond to other estimates which usually decompose aggregate output rather than per capita output.

Figure 2 and 3 about here

5.6 The Canadian dollar

Figure 1 shows that the equilibrium effective real exchange rate of the Canadian dollar depreciates continuously over most of the sample period. This finding is in line with previous studies and can be explained by a downward trend in the relative price of Canadian nontraded to traded goods (e.g. Clark and MacDonald, 2004). As in previous permanent-transitory decompositions, we find that the permanent component of the Canadian real exchange rate exhibits substantial time variability, but is more stable than the actual real exchange rate itself (Cumby and Huizinga, 1990; Clarida and Gali, 1994).

The deviations of the actual real exchange rate from its equilibrium are highly persistent, and our identification relates this low-frequency variability of the transitory component of the real exchange rate primarily to the cyclicality of the output gap rather than to inflation dynamics. Due to the strong trade and financial linkages between the Canadian and US economies, the cyclical component of the Canadian real exchange rate is captured to a large extent by developments in the multilateral value of the US dollar. Figure 1 identifies four periods of misalignment of the Canadian dollar, with significant undervaluations in the mid-1980s and the late 1990s/early 2000s, and significant overvaluations in the early 1990s and the most recent period starting in the mid-2000s.

Both periods of undervaluation of the Canadian dollar follow in the wake of multilateral appreciations of the US dollar. In the mid-1980s, the depreciation of the Canadian dollar can be associated with the appreciation of the US dollar in the wake of the Fed's monetary policy shift under Paul Volcker. The second instance of undervaluation at the end of the 1990s and the early 2000s followed a series of major currency and banking crises in Southeast Asia, Brazil, and Russia, in which US dollar denominated assets were considered to be safe investments. Similarly, the two periods of overvaluation of the Canadian real effective exchange rate are a direct consequence of the multilateral depreciations of the US dollar. In the second half of the 1980s and starting in the late 1990s, the United States experienced two periods of substantial and persistent external imbalances. Both of these episodes are associated with appreciations of all major currencies relative to the US dollar, resulting in temporary overvaluations of the Canadian dollar (Bailliu et al., 2005).[17]

6 Conclusion

This paper proposes a new approach to estimating equilibrium exchange rates for small open economies. We follow the recent literature on combining the concepts of permanent equilibrium exchange rates (PEERs) and behavioral equilibrium exchange rates (BEERs). Whereas PEERs use time-series estimators to decompose real exchange rates into their permanent and transitory

[17]The value of the Canadian dollar against the US dollar rose further with the onset of the financial crises that started in the subprime segment of the US real estate market in the summer of 2007. However, the resulting overvaluation of the Canadian dollar was short-lived as the US real effective exchange rate started to appreciate in the second half of 2008.

components, BEERs try to link the observed real exchange rate to its long-run fundamental determinants. Our analysis differs from earlier studies in two important respects. First, we derive the set of fundamentals from a well-specified open-economy model. And second, we let these fundamentals affect the exchange rate through its transitory component only, whereas both the transitory component and the long-run equilibrium real exchange rate are also driven by random determinants.

Our approach is based on a variant of the two-country open economy model of Engel and West (2006), in which the exchange rate is influenced by Taylor-rule monetary reaction functions. Explicitly taking account of the endogeneity of monetary policy may be an important missing element in traditional exchange rate models (Engel and West, 2005), and the emerging evidence on the empirical performance of Taylor rule models of the open economy is quite encouraging (Engel and West, 2006; Mark, 2009; Molodtsova and Papell, 2009).

Beside (expected) inflation and the output gap, the Engel and West (2006) model specifies the Taylor rule of one of the two countries also in terms of the real exchange rate. As this feature is frequently associated with the small country assumption, we utilize the model to identify the Canadian-US equilibrium exchange rate. Canada is an archetypical small open economy, and the Bank of Canada has traditionally engaged in exchange market management, with the bilateral Canadian-US exchange rate as the primary target of these intervention activities.

Whereas Engel and West (2006) use their model to explain the real exchange rate exclusively in terms of observable macroeconomic aggregates, we link these fundamentals to the transitory component of the exchange rate only, and let both the transitory and the long-run equilibrium real exchange rates be also influenced by random determinants. The unobserved components and the structural parameters are then jointly estimated in a Bayesian framework. In particular, the equilibrium exchange rate is explained by relative output and inflation as observable variables, and by unobserved equilibrium rates as well as unobserved transitory components in output and the exchange rate.

Using data over 1974-2009, we find that Canada's equilibrium exchange rate evolves smoothly and follows a trend depreciation. In contrast, the transitory component is found to be very persistent but much more volatile than the equilibrium rate. Whereas our results confirm a secular depreciation of the Canadian equilibrium rate also found in previous studies, we identify few but prolonged periods of currency misalignments, and associate these periods with external factors arising from shifts in the multilateral US dollar exchange rate.

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Appendices

Appendix A Technical details state space estimation

A.1 State space representation of the model in (5)-(13)

$$y_t = \begin{bmatrix} q_t & y_t & y_t^* \end{bmatrix}'; \ \alpha_t = \begin{bmatrix} \bar{q}_t & \tilde{q}'_t & \tilde{q}'_{t-1} & \bar{y}_t & \tilde{y}_t & \tilde{y}_{t-1} & \bar{y}^*_t & \tilde{y}^*_t & \tilde{y}^*_{t-1} \end{bmatrix}';$$

$$S_t = \begin{bmatrix} 0 & \lambda_t & 0 & \mu_h & 0 & 0 & \mu_f & 0 & 0 \end{bmatrix}'; \qquad \text{where } \lambda_t \equiv (\phi_1 + \phi_2)(1-\gamma_\pi)[\pi_{t-1} - \pi^*_{t-1}]$$

$$Z = \begin{bmatrix} 1 & 1 & 0 & 0 & -\xi & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad \text{where } \xi \equiv (\phi_1 + \phi_2)\gamma_y$$

$$T = \begin{bmatrix} 1 & 0 & 0 & \beta & 0 & 0 & -\beta & 0 & 0 \\ 0 & \phi_1 & \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa_1 & \kappa_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_1 & \delta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}; Q = \begin{bmatrix} \sigma^2_{\eta_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma^2_{\eta_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2_{\eta_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{\eta_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{\eta_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{\eta_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\eta_t = \begin{bmatrix} \eta_{1t} & \eta_{2t} & 0 & \eta_{3t} & \eta_{4t} & 0 & \eta_{5t} & \eta_{6t} & 0 \end{bmatrix}';$$

Modeling the dependence of $\tilde{q}_t$ on the relative output gap within the T matrix is only possible when we assume that only lagged values of the relative output gap affect $\tilde{q}_t$. An alternative is the following definition: $q_t = \bar{q}_t + \tilde{q}_t$ where $\tilde{q}_t \equiv \tilde{q}'_t - \xi(\tilde{y}_t - \tilde{y}^*_t)$ and $q'_t = \phi_1 q'_{t-1} + \phi_2 q'_{t-2} + \eta_{2t}$. This definition is possible because $(\tilde{y}_t - \tilde{y}^*_t)$ is (by construction) a stationary variable and therefore will not affect $\bar{q}_t$ as this component reflects only permanent movements of q_t. Thus the permanent component of q_t is similar to the permanent component of of $q_t - \xi(\tilde{y}_t - \tilde{y}^*_t)$.

A.2 Computational aspects of importance sampling

The idea is to use an importance density $g(\psi \mid y,x)$ as a proxy for $p(\psi \mid y,x)$, where $g(\psi \mid y,x)$ should be chosen as a distribution that can be simulated directly and is as close to $p(\psi \mid y,x)$ as possible. By Bayes' theorem and after some manipulations, equation (16) can be rewritten as

$$\overline{g} = \frac{\int g(\psi)\, z^g(\psi,y,x)\, g(\psi \mid y,x)\, d\psi}{\int z^g(\psi,y,x)\, g(\psi \mid y,x)\, d\psi}, \tag{A-1}$$

with

$$z^g(\psi,y,x) = \frac{p(\psi)\, p(y \mid \psi)}{g(\psi \mid y,x)}. \tag{A-2}$$

Using a sample of n random draws $\psi^{(i)}$ from $g(\psi \mid y,x)$, an estimate $\overline{g}_n$ of $\overline{g}$ can then be obtained as

$$\bar{g}_n = \frac{\sum_{i=1}^{n} g\left(\psi^{(i)}\right) z^g\left(\psi^{(i)},y,x\right)}{\sum_{i=1}^{n} z^g\left(\psi^{(i)},y,x\right)} = \sum_{i=1}^{n} w_i g\left(\psi^{(i)}\right), \tag{A-3}$$

with w_i

$$w_i = \frac{z^g\left(\psi^{(i)},y,x\right)}{\sum_{i=1}^{n} z^g\left(\psi^{(i)},y,x\right)}. \tag{A-4}$$

The weighting function w_i reflects the importance of the sampled value $\psi^{(i)}$ relative to other sampled values. Geweke (1989) shows that if $g(\psi \mid y,x)$ is proportional to $p(\psi \mid y,x)$, and under a number of weak regularity conditions, $\overline{g}_n$ will be a consistent estimate of $\overline{g}$ for $n \to \infty$. As an importance density $g(\psi \mid y,x)$, we take a large sample normal approximation to $p(\psi \mid y,x)$, i.e.

$$g(\psi \mid y,x) = N\left(\widehat{\psi}, \widehat{\Omega}\right) \tag{A-5}$$

where $\widehat{\psi}$ is the mode of $p(\psi \mid y,x)$ obtained from maximizing

$$\log p(\psi \mid y,x) = \log p(y \mid \psi) + \log p(\psi) - \log p(y) \tag{A-6}$$

with respect to $\widehat{\psi}$ and where $\widehat{\Omega}$ denotes the covariance matrix of $\widehat{\psi}$. Note that $p(y \mid \psi)$ is given by the likelihood function derived from the Kalman filter and we do not need to calculate $p(y)$ as it does not depend on ψ.

As any numerical integration method delivers only an approximation to the integrals in equation $(A-1)$, we monitor the quality of the approximation by estimating the probabilistic error bound for the importance sampling estimator $\overline{g}_n$ ((Bauwens et al., 1999) chap. 3, eq. 3.34). This error bound represents a 95% confidence interval for the percentage deviation of $\overline{g}_n$ from $\overline{g}$. It should not exceed 10%.

Note that the normal approximation in equation ($A-5$) selects $g(\psi \mid y, x)$ in order to match the location and covariance structure of $p(\psi \mid y, x)$ as good as possible. One problem is that the normality assumption might imply that $g(\psi \mid y, x)$ does not match the tail behavior of $p(\psi \mid y, x)$. If $p(\psi \mid y, x)$ has thicker tails than $g(\psi \mid y, x)$, a draw $\psi^{(i)}$ from the tails of $g(\psi \mid y, x)$ can imply an explosion of $z^g\left(\psi^{(i)}, y, x\right)$. This is due to a very small value for $g(\psi \mid y, x)$ being associated with a relatively large value for $p(\psi)\, p(y \mid \psi)$, as the latter is proportional to $p(\psi \mid y, x)$. Importance sampling is inaccurate in this case as this would lead to a weight w_i close to one, i.e. $\overline{g}_n$ is determined by a single draw $\psi^{(i)}$. This is signalled by instability of the weights and a probabilistic error bound that does not decrease in n. In order to help prevent explosion of the weights, we change the construction of the importance density in two respects (Bauwens et al., 1999, chap. 3). First, we inflate the approximate covariance matrix $\widehat{\Omega}$ by multiplying it by a factor of 1.1. This reduces the probability that $p(\psi \mid y, x)$ has thicker tails than $g(\psi \mid y, x)$. Second, we use a sequential updating algorithm for the importance density. This algorithm starts from the importance density defined by ($A-5$), with inflation of $\widehat{\Omega}$, estimates posterior moments for $p(\psi \mid y, x)$ and then defines a new importance density from these estimated moments. This improves the estimates for $\widehat{\psi}$ and $\widehat{\Omega}$. We continue updating the importance density until the weights stabilize. The number of importance samples n was chosen to make sure that the probabilistic error bound for the importance sampling estimator $\overline{g}_n$ does not exceed 10%.

A.3 Posterior distribution of parameter and states

An estimate $\widetilde{\psi}$ for the posterior mean $E[\psi \mid y, x]$ of the parameter vector ψ is obtained by setting $g\left(\psi^{(i)}\right) = \psi^{(i)}$ in equation ($A-3$) and taking $\widetilde{\psi} = \overline{g}_n$. An estimate $\widetilde{\alpha}_t$ for the posterior mean $E[\widehat{\alpha}_t \mid y, x]$ of the smoothed state vector $\widehat{\alpha}_t$ is obtained by setting $g\left(\psi^{(i)}\right) = \widehat{\alpha}_t^{(i)}$ in equation ($A-3$) and taking $\widetilde{\alpha}_t = \overline{g}_n$, where $\widehat{\alpha}_t^{(i)}$ is the smoothed state vector obtained from the Kalman smoother using the parameter vector $\psi^{(i)}$. In order to calculate the 10th and 90th percentiles of the posterior densities of both the parameter vector ψ and the smoothed state vector $\widehat{\alpha}_t$, let $F(\psi_j \mid y, x) = \Pr\left(\psi_j^{(i)} \leq \psi_j\right)$ with ψ_j denoting the j-th element in ψ. An estimate $\widetilde{F}(\psi_j \mid y, x)$ of $F(\psi_j \mid y, x)$ is obtained by setting $g\left(\psi^{(i)}\right) = I_j\left(\psi_j^{(i)}\right)$ in equation ($A-3$) and taking $\widetilde{F}(\psi_j \mid y, x) = \overline{g}_n$, where $I_j\left(\psi_j^{(i)}\right)$ is an indicator function which equals one if $\psi_j^{(i)} \leq \psi_j$

and zero otherwise. An estimate $\widetilde{\psi}_j^{10\%}$ of the 10th percentile of the posterior density $p(\psi \mid y, x)$ is chosen such that $\widetilde{F}\left(\psi_j^{10\%} \mid y, x\right) = 0.10$. An estimate $\widetilde{\alpha}_{j,t}^{10\%}$ of the 10th percentile of the jth element of the posterior density $p(\widehat{\alpha}_t \mid y, x)$ is obtained by setting $g\left(\psi^{(i)}\right) = \widehat{\alpha}_{j,t}^{(i)} - 1.645\sqrt{\widehat{P}_{j,t}^{(i)}}$ in equation $(A-3)$ and taking $\widetilde{\alpha}_{j,t}^{5\%} = \overline{g}_n$, where $\widehat{\alpha}_{j,t}^{(i)}$ denotes the j-th element in $\widehat{\alpha}_t^{(i)}$, and $\widehat{P}_{j,t}^{(i)}$ is the (j, j)th element of the smoothed state variance matrix $\widehat{P}_t^{(i)}$ obtained using the parameter vector $\psi^{(i)}$. The 90th percentiles are constructed in a similar way. As such the posterior distribution of the smoothed state vector $\widehat{\alpha}$ takes both parameter and filter uncertainty into account.

Appendix B Prior and Posterior parameter distributions

Figure 4 about here

Tables and Figures

Table 1: Test for structural breaks in inflation

	WD_{max}	UD_{max}	$SupF_T(1\|0)$	$SupF_T(2\|1)$	$SupF_T(3\|2)$	$SupF_T(4\|3)$	$SupF_T(5\|4)$
Canada	113.75*	97.38*	97.38*	44.46*	3.42	2.68	1.57
US	26.46*	26.46*	26.46*	9.61	1.44	0.29	0

The maximum number of breaks is set to 5. The $*$ denotes significance at the 5% level. The 5% critical values are $UD_{max} = 9.52$, $WD_{max} = 10.39$, $SupF_T(1|0) = 9.1$, $SupF_T(2|1) = 10.55$, $SupF_T(3|2) = 11.36$, $SupF_T(4|3) = 12.35$, $SupF_T(5|4) = 12.97$.

Table 2: Prior and Posterior Parameter Distributions

		Prior Distribution		Posterior Distribution	
	Parameter	Mean	90% Interval	Mean	90% Interval
Exchange Rate	ϕ_1	1.20	[1.02 , 1.38]	1.43	[1.33 , 1.54]
	ϕ_1	-0.70	[-0.88 , -0.52]	-0.47	[-0.58 , -0.36]
	γ_π	1.50	[1.28 , 1.72]	1.17	[1.07 , 1.27]
	γ_y	0.5	[0.28 , 0.72]	0.41	[0.20 , 0.62]
	β	0	[-0.13 , 0.13]	-0.02	[-0.04 , 0.00]
	$\sigma^2_{\eta_1}$	1.50	[0.49 , 2.82]	2.18	[1.07 , 3.32]
	$\sigma^2_{\eta_2}$	1.50	[0.49 , 2.82]	3.09	[1.92 , 4.34]
Output Canada	κ_1	1.20	[1.02 , 1.38]	1.45	[1.34 , 1.56]
	κ_2	-0.70	[-0.88 , -0.52]	-0.50	[-0.60 , -0.40]
	μ_h	0.45	[0.32 , 0.58]	0.46	[0.39 , 0.53]
	$\sigma^2_{\eta_3}$	0.5	[0.33 , 0.69]	0.49	[0.37 , 0.62]
	$\sigma^2_{\eta_4}$	0.5	[0.33 , 0.69]	0.44	[0.32 , 0.57]
Output US	κ_1	1.20	[1.02 , 1.38]	1.47	[1.36 , 1.57]
	κ_2	-0.70	[-0.88 , -0.52]	-0.52	[-0.62 , -0.42]
	μ_f	0.45	[0.32 , 0.58]	0.46	[0.40 , 0.52]
	$\sigma^2_{\eta_5}$	0.5	[0.33 , 0.69]	0.30	[0.22 , 0.38]
	$\sigma^2_{\eta_6}$	0.5	[0.33 , 0.69]	0.34	[0.25 , 0.42]

The prior distribution is assumed to be Gaussian for all elements in ψ, except the variance parameters which are assumed to be gamma distributed. With n=50,000 for the initial importance function and all updates, the probabilistic error bound for the importance sampling estimator g_n is well below 10% for all coefficients. The number of subsequent updates of the importance density is 3 (see Appendix A for details).

Figure 1: Equilibrium and transitory exchange rate

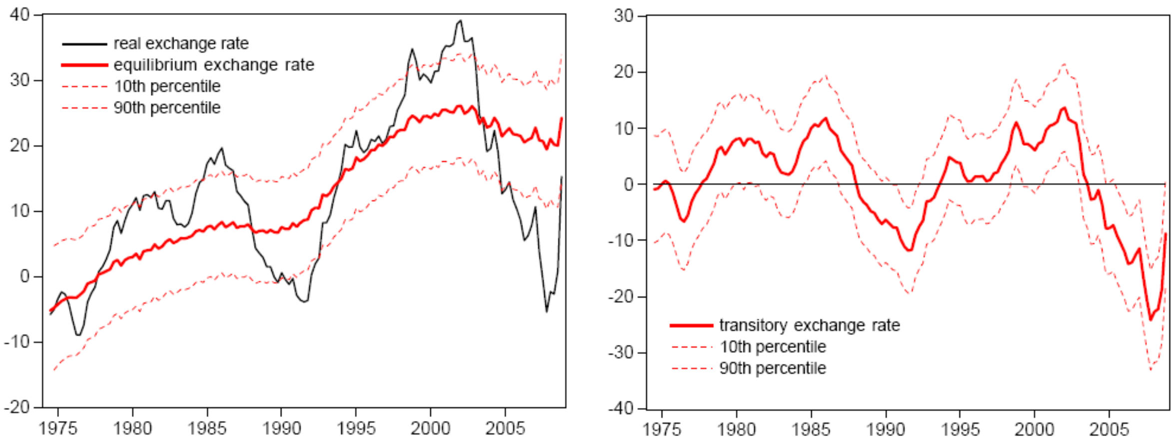

Figure 2: Canada: Output Gap and Potential Output

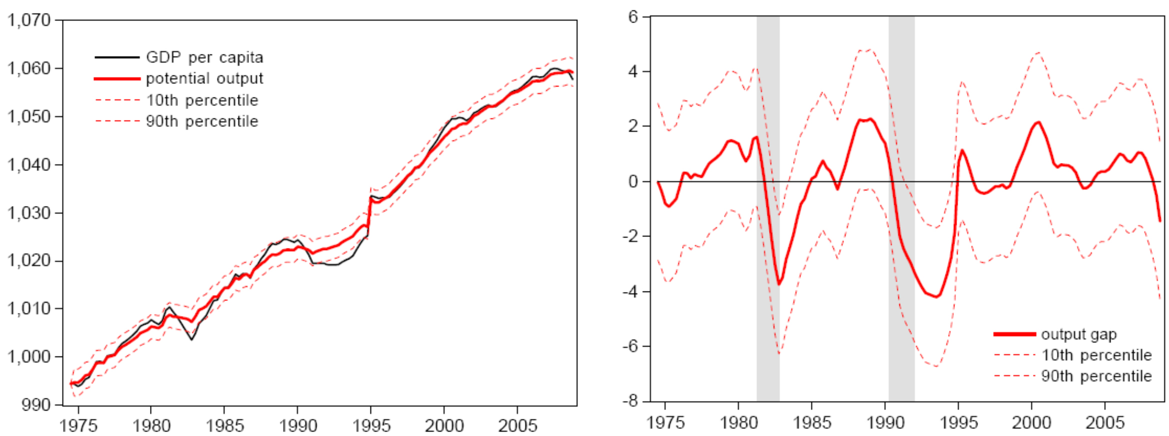

Figure 3: US: Output Gap and Potential Output

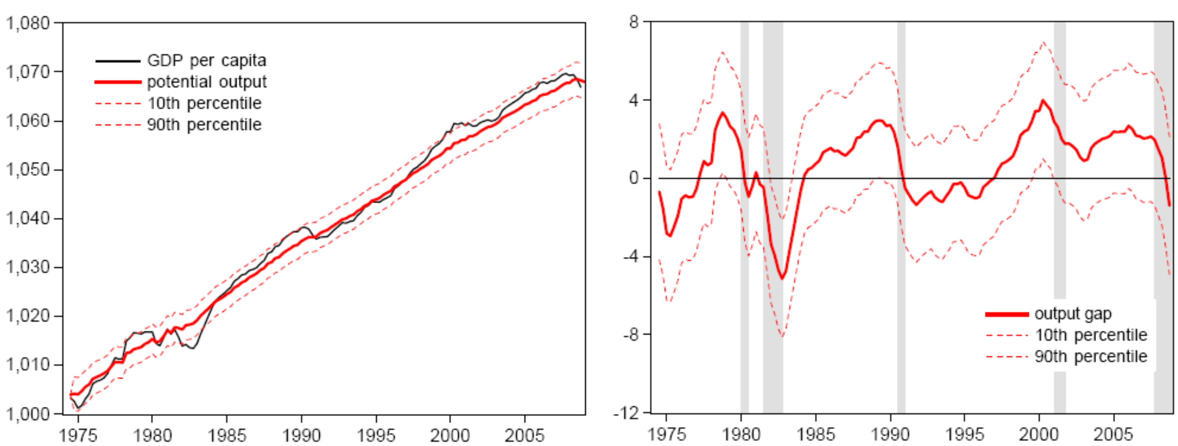

Figure 4: Prior and Posterior parameter distributions