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WORKING PAPER

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Abstract

We develop a directional measure of income mobility. This measure can be expresssed as a rank dependent mean of the individual mobilities in society and allows to give more weight to lower individual mobilities. The class includes the measures of directional mobility encountered in the literature. We apply our measure to compare income mobility between the United States and Germany, and find that giving more weight to the bottom end of the mobility distribution reverses the mobility ranking.
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1 Motivation

This note contributes to a small but rapidly expanding literature on the axiomatic characterization of income mobility measures. An income mobility measure evaluates the change from an initial to a final income distribution. Often, as in this paper, it is convenient to draw a distinction between aggregate income mobility, which measures the mobility for a country or society as a whole, and individual income mobility, which is the mobility encountered by a single individual. By definition, individual mobility depends only on the individual's income in the initial and final period. A widely accepted and uncontroversial assumption in the literature on axiomatic mobility measurement is the condition of Weak Decomposability. This condition states that aggregate mobility can be written as an increasing function of the individual mobilities alone. Once this is accepted, the specific form of the aggregate mobility measure can be established by solving two remaining issues. The first concerns the particular functional form for the individual mobility measure. As this functional form depends only on two variables —initial and final income— it can be quickly characterized by combining certain elementary structural axioms.

The second issue concerns the particular way in which individual mobilities are aggregated. The most straightforward way is to take the average over all individual mobilities. This averaging is the procedure that is adopted most frequently in the literature (see, among others, Cowell (1985), Fields and Ok (1996), Fields and Ok (1999), Mitra and Ok (1998), Schluter and Van de gaer (2010), D'Agostino and Dardanoni (2009b) and Tsui (2009)).

Many mobility concepts exist in the literature - see, e.g., Fields (2007). We axiomatize a directional income mobility measure where the aggregation procedure depends on the rank of the individual mobilities[1]. As an illustration, consider an example of a society with three individuals, an initial income distribution $\mathbf{x} = (10, 10, 10)$ and two final income distributions $\mathbf{y} = (10, 20, 30)$ and $\mathbf{y}' = (20, 20, 20)$. Furthermore, assume that we measure individual income mobility by ratio of final to initial period income, such that the measure is increasing in income growth. Then the process whereby the income vector $\mathbf{x}$ changes into $\mathbf{y}$, denoted as $\mathbf{x} \to \mathbf{y}$, gives the vector of individual mobilities $(1, 2, 3)$ and the process $\mathbf{x} \to \mathbf{y}'$ gives the individual mobility vector $(2, 2, 2)$. If aggregate mobility is taken to be the unweighted average of the individual mobilities then the two processes $\mathbf{x} \to \mathbf{y}$ and $\mathbf{x} \to \mathbf{y}'$ should have the same aggregate mobility. On the other hand, it may be argued that the mobility present in the process $\mathbf{x} \to \mathbf{y}'$ is more desirable than the mobility in the process $\mathbf{x} \to \mathbf{y}$ because in the former the individual mobilities are more equally distributed than in the latter. It seems intuitive that the aggregation procedure should be able to take this into account.

In this paper, we develop a rank dependent mobility measure which allows that individuals with a lower individual mobility receive a higher weight in the aggregation of the vector of individual mobilities. This corresponds to a notion of aversion towards inequality in individual mobilities: individuals with the lowest mobility should receive the highest weight in the aggregation procedure. It is an attractive property when individual mobility measures something that is desirable, such as income growth in the example above. For

[1]The approach is distinct from D'Agostino and Dardanoni (2009a). They develop rank mobility comparisons based only on the information on the order of initial and final incomes. It also differs from Van Kerm (2009) who depicts individual mobilities as a function of individuals' rank in the initial distribution of income and proposes to measure aggregate mobility by giving weights dependent on individuals' rank in the intitial distribution of income. This allows him to give a greater weight to the income mobility of the initially poor but violates our weak decomposability axiom.

cases where the mobility index is undirectional, i.e. when the individual mobility index makes no difference between an equally sized increase and decrease in incomes (see Fields and Ok (1996), Fields and Ok (1999) and D'Agostino and Dardanoni (2009b)), this issue seems less relevant.

Hence we first characterize two directional measures of individual mobility. Both measures are scale invariant: they are unchanged when the individual's initial and final income are multiplied by the same positive scalar (axiom SI). They are directional: they are increasing in final and decreasing in initial income (axiom M). Finally they are path independent: one measure satisfies multiplicative path independence (axiom MPI), the other additive path independence (API). The former (latter) means that when moving first from an initial to an intermediate and then to a final income level, the mobility in moving from the initial to the final income level can be written as a multiplicative (additive) function of the mobility in the transition from the initial to the intermediate and the mobility of the intermediate to the final income level.

Next, we move to the main contribution of this paper: the characterization of the rank dependent aggregation procedure. We use a framework that is similar in spirit to Bossert (1990)'s characterization of the S-Gini index. To apply it to the mobility framework, we add the requirement of Weak Decomposability (axiom WD), explained above. Bossert's first structural assumption applied to the mobility context becomes then that the aggregate mobility measure must be both relative and absolute with respect to individual mobilities (axioms RI and TI). His second structural assumption becomes that the mobility measure has to satisfy an aggregation requirement: for a population of size n, the income mobility of the population depends on the income mobility of the group of $n-1$ most mobile members in the society and the mobility of the least mobile individual (axiom D-HM). Perhaps these requirements are less intuitive in the mobility context than in the context of inequality measurement, but most mobility measures that have been proposed in the literature satisfy them. Beside these structural axioms we also impose two normative axioms. The first axiom, population invariance, implies that any k-fold reproduction of the society should leave aggregate mobility unchanged (axiom PI). The last axiom, priority for lower mobilities, states that aggregate mobility increases more when additional mobility is allocated to individuals with lower individual mobility than when it is allocated to individuals with higher individual mobility (axiom PLM).

The combination of these axioms leads, in the Donaldson and Weymark (1980) terminology, to a family of single-series Gini indices defined over two possible measures of individual mobility. The first yields a generalization of a measure proposed by Schluter and Van de gaer (2010), the second a generalization of a measure developed by Fields and Ok (1999). We illustrate our measure by applying it to the CNEF-adjusted data sets of the PSID (United States) and the GSOEP (Germany) surveys. We show that the ranking between the two countries depends crucially on the value of a parameter which controls the degree to which more weight is given at the bottom of the individual mobility distribution.

In section 2, we provide our characterization and section 3 gives the empirical illustration. Section 4 concludes.

2 Characterization

Consider a set of individuals $\{1, \dots, n\}$ and two income distributions $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{++}$. Individual i's initial income is given by x_i while his final income is given by y_i. We measure aggregate mobility of going from distribution $\mathbf{x}$ to distribution $\mathbf{y}$ by a real valued continuous function $M^n(\mathbf{x},\mathbf{y})$. In particular, given the income distribution vectors $\mathbf{x},\mathbf{x}',\mathbf{y}$ and $\mathbf{y}'$, we have that $M^n(\mathbf{x},\mathbf{y}) \geq M^n(\mathbf{x}',\mathbf{y}')$ if the process $\mathbf{x} \to \mathbf{y}$ is deemed to have more mobility than the process $\mathbf{x}' \to \mathbf{y}'$. For a scalar $x \in \mathbb{R}_+$ we write $x \cdot \mathbf{1}$ to indicate the n-dimensional vector $(x, x, \ldots, x)$. We begin by imposing some properties on the measure of individual mobility, M^1. Our first property states that individual mobility should be increasing in final period income and decreasing in initial period income.

Axiom M (**Monotonicity):** *for all $x, y, x', y' \in \mathbb{R}_{++}$, if $x \leq x'$ and $y \geq y'$, then:*

$$M^1(x,y) \geq M^1(x',y) \text{ and } M^1(x,y) \geq M^1(x,y').$$

This axiom clearly shows that our measure will be directional, i.e. we make a distinction between good or upward and bad or downward mobility. While intuitive and accepted in questionnaire studies on the measurement and evaluation of mobility (see Bernasconi and Dardanoni (2005)), this idea has only been used in the axiomatic literature on mobility measurement by Schluter and Van de gaer (2010), where it was directly imposed on $M^n(\mathbf{x},\mathbf{y})$ instead of on $M^1(x,y)$ to characterize a measure of upward structural mobility.

Our second condition says that individual mobilities should remain invariant when both initial and final income are scaled up or down with a common factor. This condition is important when comparing individual mobilities for countries using different currencies.

Axiom SI **(Scale Invariance):** *for all $x, y \in \mathbb{R}_{++}$ and $\lambda > 0$:*

$$M^1(x,y) = M^1(\lambda x, \lambda y).$$

The condition of scale invariance allows us to define a real valued function f such that, $M^1(x,y) = M^1(1, y/x) = f(y/x)$. In order to pinpoint the functional form of f, we need an additional condition. We choose to impose a path independency property. Consider three periods. An individual's first period income x changes to y in the second period and to z in the third. Our path independence axiom states that the individual's mobility, from the first to the third period (x to z) can be written as a function of the two single period mobilities (x to y and y to z). In its most general form, it requires the existence of a function G such that $G(M^1(x,y), M^1(y,z)) = M^1(x,z)$. We choose two particular forms for this function G[2].

Axiom MPI **(Multiplicative Path Independence):** *for all $x, y, z \in \mathbb{R}_{++}$:*

$$M^1(x,z) = M^1(x,y).M^1(y,z).$$

Axiom API **(Additive Path Independence):** *for all $x, y, z \in \mathbb{R}_{++}$:*

$$M^1(x,z) = M^1(x,y) + M^1(y,z).$$

One can easily verify that MPI requires the function f to satisfy Cauchy's fourth functional equation while API requires f to satisfy Cauchy's third functional equation. This gives the following partial result (see, for example Aczél (1966), p.39):

[2]Observe that the mere existence of G does not impose any additional requirements on f. Indeed, one can always choose $G(a,b) = g^{-1}(g(a) + g(b))$ with $g = \ln \circ f^{-1}$.

Lemma 1

- *M^1 satisfies M, SI and MPI if and only if $M^1(x, y) = (y/x)^r$ for some $r > 0$.*
- *M^1 satisfies M, SI and API if and only if $M^1(x, y) = r \ln(y/x)$ for some $r > 0$.*

Our next axiom states that total mobility should only depend on the values of the individual mobilities and be increasing in individual mobilities. Its interpretation was already given in the introduction.

Axiom WD **(Weak Decomposability):** *for all $n \in \mathbb{N}$ and all $\mathbf{x}, \mathbf{y}, \mathbf{x}'$ and $\mathbf{y}' \in \mathbb{R}^n_{++}$, if for all $i \in \{1, \ldots, n\}$ and $i \neq j$,*

$$M^1(x_i, y_i) = M^1(x'_i, y'_i),$$

then

$$M^n(\mathbf{x}, \mathbf{y}) \geq M^n(\mathbf{x}', \mathbf{y}') \text{ if and only if } M^1(x_j, y_j) \geq M^1(x'_j, y'_j).$$

Given two income distributions $\mathbf{x}$ and $\mathbf{y}$ and an individual mobility measure M^1, we may construct the vector of individual mobilities $\mathbf{m}$, determined by its elements $m_i = M^1(x_i, y_i)$. The following is an immediate consequence of axiom WD.

Lemma 2 *If M^n satisfies WD, then there exist strictly increasing and continuous functions W^n such that for all $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n_{++}$:*

$$M^n(\mathbf{x}, \mathbf{y}) = W^n(\mathbf{m}).$$

Lemma 2 shows that we may restrict ourselves to the ranking of all vectors of individual mobilities $\mathbf{m} \in \mathbb{R}^n_{++}$. For each $n \in \mathbb{N}$, consider the function $\varepsilon^n : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ such that:

$$W^n(\mathbf{m}) = W^n(\varepsilon^n(\mathbf{m}) \cdot \mathbf{1}).$$

The function ε^n is similar to the equally distributed equivalent income that is well known from the literature on inequality measurement (see Atkinson (1970)). It is the amount of individual mobility, which if distributed equally to everyone would render aggregate mobility equal to the case where the individual mobility vector is equal to $\mathbf{m}$. The 'greater than or equal' ordering implied by ε^n coincides with the ordering derived from W^n. This follows from the monotonicity of the function W^n:

$$\begin{aligned} & W^n(\mathbf{m}) \geq W^n(\mathbf{m}') \\ \Longleftrightarrow\ & W^n(\varepsilon^n(\mathbf{m}) \cdot \mathbf{1}) \geq W^n(\varepsilon^n(\mathbf{m}') \cdot \mathbf{1}) \\ \Longleftrightarrow\ & \varepsilon^n(\mathbf{m}) \geq \varepsilon^n(\mathbf{m}'). \end{aligned}$$

As such, we may proceed by imposing additional restrictions upon the function ε^n instead of upon the function W^n. Observe that for all $\mathbf{m} \in \mathbb{R}_+$:

$$W^n(m \cdot \mathbf{1}) = W^n(\varepsilon^n(m \cdot \mathbf{1}) \cdot \mathbf{1}).$$

This implies that $\varepsilon^n(m \cdot \mathbf{1}) = m$ for all values of m and n.

Our following axiom states that comparisons between mobilities remain invariant under a common multiplication of the individual mobilities. In other words, the ranking derived from the aggregate mobility index should not depend on the particular units in which the individual mobilities are measured.

Axiom RI **(Relative Invariance):** *for all* $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n_+$ *and* $\lambda > 0$*,*

$$\textit{if } W^n(\mathbf{r}) = W^n(\mathbf{s}) \textit{ then } W^n(\lambda\mathbf{r}) = W^n(\lambda\mathbf{s}).$$

The most important implication of *RI* is that the function ε^n becomes homogeneous of degree one. Indeed, from the definition of ε^n, we have that $W^n(\mathbf{m}) = W^n(\varepsilon^n(\mathbf{m}) \cdot \mathbf{1})$ such that, by RI

$$W^n(\lambda\mathbf{m}) = W^n(\lambda\varepsilon^n(\mathbf{m}) \cdot \mathbf{1}).$$

From the definition of ε^n we also have that,

$$W^n(\lambda\mathbf{m}) = W^n(\varepsilon^n(\lambda\mathbf{m}) \cdot \mathbf{1}).$$

Combining the last two equalities, we get that $W^n(\lambda\varepsilon^n(\mathbf{m}) \cdot \mathbf{1}) = W^n(\varepsilon^n(\lambda\mathbf{m}) \cdot \mathbf{1})$, from which since W^n is monotonic

$$\lambda\varepsilon^n(\mathbf{m}) = \varepsilon^n(\lambda\mathbf{m}).$$

The next axiom states that comparisons between mobilities remains invariant under a common translation of the individual mobilities. In other words, the ranking derived from the aggregate mobility indices should not depend on the particular origin that is chosen for the measurement of the individual mobilities.

Axiom TI **(Translation Invariance):** *for all* $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n_+$ *and* $\lambda > 0$*,*

$$\textit{if } W^n(\mathbf{r}) = W^n(\mathbf{s}) \textit{ then } W^n(\mathbf{r} + \lambda \cdot \mathbf{1}) = W^n(\mathbf{s} + \lambda \cdot \mathbf{1}).$$

The axiom of Translation Invariance imposes that ε^n is independent of origin. From the definition of ε^n, we have $W^n(\mathbf{m}) = W^n(\varepsilon^n(\mathbf{m}) \cdot \mathbf{1})$, such that, by TI

$$W^n(\mathbf{m} + \lambda \cdot \mathbf{1}) = W^n(\varepsilon^n(\mathbf{m}) \cdot \mathbf{1} + \lambda \cdot \mathbf{1}) = W^n((\varepsilon^n(\mathbf{m}) + \lambda) \cdot \mathbf{1}).$$

At the same time, from the definition of ε^n,

$$W^n(\mathbf{m} + \lambda \cdot \mathbf{1}) = W^n\left(\varepsilon^n(\mathbf{m}{+}\lambda \cdot \mathbf{1}) \cdot \mathbf{1}\right),$$

such that the combination of the last two equations yields, because of the monotonicity of W^n,

$$\varepsilon^n(\mathbf{m} + \lambda \cdot \mathbf{1}) = \varepsilon^n(\mathbf{m}) + \lambda.$$

Conditions RI and TI impose a very specific functional for $\varepsilon^2(m_1, m_2)$.

Lemma 3 *The function* W^2 *satisfies WD, TI and RI, if and only if there exist numbers* γ_1^2 *and* $\gamma_2^2 \in [0, 1]$*, such that:*

$$\gamma_1^2 + \gamma_2^2 = 1 \textit{ and } \varepsilon^2(m_1, m_2) = \gamma_1^2 m_1 + \gamma_2^2 m_2.$$

Lemma 3 shows that for populations of two individuals, aggregate mobility can be written as a weighted sum of individual mobilities. In order to derive the functional form for societies with more than two individuals we need an additional axiom.

For any vector $\mathbf{m} \in \mathbb{R}^n_+$, let $\widetilde{\mathbf{m}}$ be a permutation of $\mathbf{m}$ such that $m_1 \geq m_2 \ldots \geq m_n$. Our next axiom states that aggregate mobility only depends on the mobility of the individual with the lowest level of individual mobility and on the aggregate mobility of the $n-1$ other individuals. Although it may seem like a strong restriction, we should note that it is much weaker than most of the decomposability axioms in the literature. See, for example, Fields and Ok (1996) axiom 2.4, Fields and Ok (1999) subgroup decomposability, Mitra and Ok (1998) axiom PC, Schluter and Van de gaer (2010) subgroup consistency, D'Agostino and Dardanoni (2009b) subvector consistency.

Axiom D-HM **(Decomposability with respect to Highest Mobility):** *for all $n \in \mathbb{N}$ and all $\mathbf{m}, \mathbf{m}' \in \mathbb{R}^n_+$,*

$$\textit{if } W^{n-1}(\tilde{m}_1, \ldots, \tilde{m}_{n-1}) = W^{n-1}(\tilde{m}'_1, \ldots, \tilde{m}'_{n-1}) \textit{ and } \tilde{m}_n = \tilde{m}'_n,$$
$$\textit{then } W_n(\mathbf{m}) = W^n(\mathbf{m}').$$

Together with TI and RI, we can derive the following partial result:

Lemma 4 *The function M^n satisfies WD, TI and RI and D-HM, if and only if there exist positive numbers $\gamma_1^n, \ldots, \gamma_n^n$ summing to one, such that:*

$$\varepsilon^n(\mathbf{m}) = \sum_{i=1}^{n} \gamma_i^n \tilde{m}_i.$$

Our next axiom is known as population invariance or replication invariance. It says that a k-fold replication of the population does not change aggregate mobility.

Axiom PI **(Population Invariance):** *for all $n, k \in \mathbb{N}$ and all $\mathbf{m} \in \mathbb{R}^n_+$,*

$$W^n(\mathbf{m}) = W^{kn}(\underbrace{\mathbf{m}, \mathbf{m}, \ldots, \mathbf{m}}_{k times}).$$

PI allows us to determine the functional form of the coefficients γ_i. Indeed, theorems 1 and 2 of Donaldson and Weymark (1980) show that PI imposes that there exist a $\delta \in \mathbb{R}_{++}$ such that for all $i \in \mathbb{N}$,

$$\gamma_i^n = (i^\delta - (i-1)^\delta)/n^\delta.$$

Hence,

Lemma 5 *The function M^n satisfies WD, TI, RI , D-HM and PI if and only if there exists a number δ such that:*

$$\varepsilon^n(\mathbf{m}) = \frac{\sum_{i=1}^{n}(i^\delta - (i-1)^\delta)\tilde{m}_i}{n^\delta}.$$

Finally, we introduce one additional axiom. This axiom states that an allocation of additional mobility increases mobility more if it is allocated to an individual with lower mobility.

Axiom PLM **(Priority for Lower Mobilities):** *for all $x \in \mathbb{R}_+$, $\mathbf{m} \in \mathbb{R}^n_+$ and $\sigma > 0$, if $m_i < m_j$, then:*

$$W^n(m_1, \dots, m_i + \sigma, \dots, m_j, \dots, m_n) \geq W^n(m_1, \dots, m_i, \dots, m_j + \sigma, \dots, m_n).$$

One immediately verifies that PLM imposes the condition that $\delta \geq 1$. Using previous lemmata, the following proposition is straightforward:

Proposition 1 *For all $n \in \mathbb{N}$ if M^n satisfies M, SI, MPI, RI, TI, D-HM, PI and PLM, then there exist a $\delta \geq 1$ and a number $r > 0$, such that:*

$$M^n(\mathbf{x}, \mathbf{y}) = \frac{1}{n^\delta} \sum_{i=1}^{n} (i^\delta - (i-1)^\delta)\ \tilde{m}_i.$$

with $\tilde{m}_i$ the individual mobilities m_i ranked increasingly, where

$$m_i = M^1(x_i, y_i) = (y_i/x_i)^r.$$

If API is satisfied instead of MPI, then,

$$m_i = M^1(x_i, y_i) = r \ln(y_i/x_i).$$

The first measure of Proposition 1, for $\delta = 1$, reduces to the measure proposed in Schluter and Van de gaer (2010),

$$M_{SV} = \frac{1}{n} \sum_{i=1}^{n} (y_i/x_i)^r\ .$$

Here the parameter r is a sensitivity parameter: higher values of r lead to larger differences in individual mobilities without changing their ranking. Especially the relative size of high values of y_i/x_i increases rapidly as r increases. The second measure of Proposition 1, for $\delta = 1$, reduces to the measure of Fields and Ok (1999):

$$M_{FO} = \frac{1}{n} \sum_{i=1}^{n} r \ln (y_i/x_i)\ .$$

The value of r does not change the ranking when comparing aggregate mobility in two situations, and so r can be put equal to 1. These two measures compute aggregate mobility by taking the unweighted sum of all individual mobilities. In the literature on mobility measurement the notion of exchange mobility takes an important place. It requires that covariance decreasing income swaps in either the initial or the final period increase mobility [3]. Measure M_{SV} satisfies this notion, while measure M_{FO} is insensitive to covariance decreasing income swaps.

Since $\delta \geq 1$ in proposition 1, our measure generalizes the measures M_{SV} and M_{FO}. For $\delta > 1$ it gives more weight to individual mobilities at the bottom of the individual mobility distribution. Naturally, this increased generality comes at a cost, as the generalizations of

[3]See Tsui (2009) for various equivalent ways to define exchange mobility.

both M_{SV} and M_{FO} are sensitive to covariance decreasing income swaps, but in a non-trivial way: covariance decreasing income swaps may actually decrease the value of this mobility measure. In addition, contrary to measures M_{SV} and M_{FO}, our mobility measure is no longer additively decomposable in the mobilities of subgroups of the population. Yet, we believe that the possibility to attach more weight to individual mobilities at the bottom of the individual mobility distribution is worth this cost. In the next section, we'll see that it can reverse the mobility ranking between countries.

3 Empirical illustration

For the remaining part of this section it is more convenient to define the vector $\hat{m}$ by $\hat{m}_i = \tilde{m}_{n-i+1}$. In other words, $\hat{m} = (\hat{m}_1, \ldots, \hat{m}_n)$ is a permutation of $m = (m_1, \ldots, m_n)$ such that $\hat{m}_1 \leq \ldots \leq \hat{m}_n$. Then we can write:

$$M^n(\mathbf{x}, \mathbf{y}) = \frac{1}{n^\delta} \sum_{i=1}^{n} \left((n-i+1)^\delta - (n-i)^\delta \right) \hat{m}_i.$$

For our empirical illustration, we first construct a continuous analogue of M^n. Consider the discrete difference $\Delta(i^\delta) = i^\delta - (i-1)^\delta$, such that

$$M^n(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \hat{m}_i \Delta \left(\left(1 - \frac{i-1}{n} \right)^\delta \right).$$

Obviously, $(i-1)/n$ corresponds to the fraction of individuals that have mobility less than the individual with mobility equal to the i-th value in the vector $(\hat{m}_1, \ldots, \hat{m}_n)$. A straightforward limiting value can be obtained by taking the limits for n to infinity and taking $F(\hat{m}_i) \approx (i-1)/n$:

$$M^\infty(F) = \int_0^1 x \, d((1-F(x)^\delta) = \int_0^1 \delta \, x(1-F(x))^{\delta-1} \, dF(x).$$

Here, the function $F(x)$ is the continuous cumulative distribution function of the individual mobilities. The advantage of the measure $M^\infty(F)$ is that it is independent of the population size, hence it can be used to compare two populations across different time periods or different countries. A natural estimator of the statistic $M^\infty(F)$ based on a finite i.i.d. sample of size n would be the measure M^n. Unfortunately, this estimator is biased and, in general, its bias depends on the sample size n, the parameter δ and the distribution F.

As shown by Demuynck (2009), for integer values of δ, the following estimator is an unbiased estimator for the index $M^\infty(F)$:

$$\mathcal{M}^n(\mathbf{m}) = \sum_{i=1}^{n} a_i \hat{m}_i,$$

with $\mathbf{m}$ an i.i.d. sample of size n from the distribution F and:

$$a_1 = \delta/n \qquad \text{and} \qquad a_i = a_{i-1} \left(1 - \frac{(\delta-1)}{n-(i-1)} \right) \text{ for } i > 1.$$

This is the estimator used in the empirical analysis. We compare aggregate income mobility between Germany and the United states using the CNEF adjusted data sets of the GSOEP (Germany) and the PSID (United States) surveys. These data sets are widely used in other studies of income mobility (see, for example Burkhauser and Poupore (1997), Maasoumi and Trede (2001), Schluter and Trede (2003) and Trede (1998)). As customary, we take the post government net incomes, deflated and truncated at the 1 and 99 percent quantiles. The resulting sample contains more than 15.000 observations. Differences in household size are taken into account by the OECD equivalence scale which divides the incomes by the square root of household size. The finite sample distribution of $\mathcal{M}^n$ is unknown. Therefor, we estimated all confidence intervals by a nonparametric bootstrapping procedure of 20.000 resamples. We restrict ourselves to the individual measures $M^1(x_i, y_i) = (y_i/x_i)^r$, ($r = 0.2, 0.5, 1, 1.5$ and 2) and $M^1(x_i, y_i) = \ln(y_i/x_i)$. We further restrict ourselves to the years 1984/85 and 1996/97 but the results for other years are similar. Table 1 provides the result of the statistical test whether income mobility in the US is larger than in Germany or not[4].

TABLE 1 AROUND HERE

We see that for $\delta = 1$ the US is always more mobile than Germany. This is the case where our measure coincides with the measures of Schluter and Van de gaer (2010) and Fields and Ok (1999) discussed at the end of the previous section. However, as δ increases, Germany becomes more mobile than the US. For the individual mobility measure $(y_i/x_i)^r$, how much δ has to increase for the ranking to reverse depends on the value of r. Larger values for r do not change the ranking of individual mobilities, but increases the difference between high and low individual mobilities, such that in the comparison between the US and Germany a greater weight has to be given to low individual mobilities before the ranking reverses.

Clearly, for our measures it makes a big difference whether individual mobilities are weighted or not before aggregation. Our procedure (with $\delta > 1$) gives more weight to the mobilities at the lower end of the mobility distribution. Hence our result indicates that those at the bottom end of the mobility distribution in the US face a larger percentage decrease in their incomes than those that are at the same percentile of the mobility distribution in Germany. This is confirmed by the plots of the cumulative distribution functions of (y_i/x_i) in figure 1 below and was already observed by Chen (2009), who considers 5 year income movements for a set of countries including the US and Germany -see figure 4 p.85 and the discussion following it. Since the cumulative distribution functions do cross, there is no first order dominance of one distribution over the other. Hence, the mobility judgment depends on the way individual mobilities are aggregated. Our measures are the first to make this explicit. They allow us to conclude that, even though average individual mobilities are higher in the US, a moderate concern with the distribution of individual mobilities allows one to conclude that Germany is more mobile than the US.

FIGURE 1 AROUND HERE

[4]The mobility numbers themselves are not informative, as mobility numbers provide ordinal information only.

4 Conclusion

We argue that the standard practice of simply adding individual mobility numbers to obtain an aggregate mobility number can be questioned in contexts where mobility is supposed to measure something that is desirable for individuals, like income growth. This leads us to axiomatize rank dependent measures of upward structural mobility, such that more weight can be given to lower individual mobilities. The result is a generalization of the unweighted measures of upward structural mobility proposed by Fields and Ok (1999) and Schluter and Van de gaer (2010). Observe, however that the aggregation procedure can, in principle, be applied to other measures of individual mobility than those presented here.

Our empirical illustration shows that the issue is very relevant in the comparison of income mobilities between the US and Germany: with unweighted individual mobilities the US is more mobile than Germany, but the ranking reverses when more weight is given to lower individual mobilities than to higher individual mobilities. We therefore conclude that the issue raised in this paper is of importance when comparing mobilities between countries.

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A Proofs

proof of lemma 3 Consider $\mathbf{m} = (m_1, m_2)$ and assume wlog that $m_1 \geq m_2$ then:

$$\begin{aligned}
&\text{(TI)} & \varepsilon^2(m_1, m_2) &= \varepsilon^2(m_1 - m_2, 0) + m_2 \\
&\text{(RI)} & &= \varepsilon^2(1, 0)(m_1 - m_2) + m_2 \\
& & &= \varepsilon^2(1, 0)m_1 + (1 - \varepsilon^2(1, 0))m_2,
\end{aligned}$$

Now, set $\gamma_1^2 = \varepsilon^2(1, 0)$ and set $\gamma_2^2 = (1 - \varepsilon^2(1, 0))$. By WD:

$$0 = \varepsilon^2(0, 0) \leq \varepsilon^2(1, 0) \leq \varepsilon^2(1, 1) = 1.$$

Hence, both γ_1^2 and γ_2^2 are positive.

proof of lemma 4 Observe that Axiom D-HM allows the existence of a two placed function L^n such that:

$$\varepsilon^n(\tilde{m}_1, \dots, \tilde{m}_n) = L^n(\varepsilon^{n-1}(\tilde{m}_1, \dots, \tilde{m}_{n-1}), \tilde{m}_n).$$

The proof of the lemma is by induction. Lemma 3 gives the proof for $n = 2$. Now, assume that it holds up to $n-1$ and let us show that the result holds for n. Then:

$$\begin{aligned}
&\text{(D-HM)} && \varepsilon^n(\tilde{m}_1, \dots, \tilde{m}_n) = L^n(\varepsilon^{n-1}(\tilde{m}_1, \dots, \tilde{m}_{n-1}), \tilde{m}_n) \\
&\text{(TI)} && = L^n(\varepsilon^{n-1}(\tilde{m}_1 - \tilde{m}_n, \dots, \tilde{m}_{n-1} - \tilde{m}_n), 0) + \tilde{m}_n \\
& && = L^n(\varepsilon^{n-1}(\mathbf{1} \cdot \varepsilon^{n-1}(\tilde{m}_1 - \tilde{m}_n, \dots, \tilde{m}_{n-1} - \tilde{m}_n)), 0) + \tilde{m}_n \\
&\text{(RI)} && = L^n(\varepsilon^{n-1}(\mathbf{1}), 0) \cdot \varepsilon^{n-1}(\tilde{m}_1 - \tilde{m}_n, \dots \tilde{m}_{n-1} - \tilde{m}_n)) + \tilde{m}_n \\
&\text{(TI)} && = L^n(1, 0) \cdot (\varepsilon^{n-1}(\tilde{m}_1, \dots, \tilde{m}_{n-1}) - \tilde{m}_n) + \tilde{m}_n \\
& && = L^n(1, 0)\varepsilon^{n-1}(\tilde{m}_1, \dots, \tilde{m}_{n-1}) + (1 - L^n(1, 0))\tilde{m}_n.
\end{aligned}$$

Now, substituting $\varepsilon^{n-1}(\tilde{m}_1, \dots, \tilde{m}_{n-1}) = \sum_{i=1}^{n-1} \gamma_i^{n-1} \tilde{m}_i$ and defining for $i < n$,

$$\gamma_i^n = \gamma_i^{n-1} L^n(1, 0)$$

and for $i = n$,

$$\gamma_n^n = (1 - L^n(1, 0)),$$

we derive the expression:

$$\varepsilon^n(\widetilde{\mathbf{m}}) = \sum_{i=1}^{n} \gamma_i^n \tilde{m}_i.$$

It is easy to see that $\sum_{i=1}^{n} \gamma_i^n = 1$ and that all terms are positive.

Table 1: Comparison of mobility between Germany and US

		r					
1984/85	log	.2	.4	.7	1	1.5	2
δ				$M_\infty^{PSID} > M_\infty^{GSOEP}$			
1	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
2	FALSE^{ns}	FALSE^{ns}	TRUE^{ns}	TRUE^{ns}	TRUE	TRUE	TRUE
4	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE^{ns}
6	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE
8	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE

		r					
1996/97	log	.2	.4	.7	1	1.5	2
δ				$M_\infty^{PSID} > M_\infty^{GSOEP}$			
1	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
2	FALSE	FALSE	FALSE	FALSE	FALSE^{ns}	TRUE	TRUE
4	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE
6	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE
8	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE

NOTE: All signs execpt the ones with ns as supperscript are significant at the 95% level. Confidence intervals are constructed using nonparametric bounds based on 20.000 bootstrap resamples.

Figure 1: Cumulative distribution function of individual mobilities 1984/85 and 1996/97.