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WORKING PAPER

Common ordering extensions [1]

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Abstract

This article provides necessary and sufficient conditions for a collection of binary relations to have a common ordering extension. We also characterize the quasi-ordering that is obtained by taking the intersection over all these ordering extensions. Next, we consider the special case where the collection contains only two relations. In this special case, our necessary and sufficient conditions can be reformulated to include solely binary relations that are defined on a certain subset of the universal domain. The usefullness of our results are illustrated with several examples and we relate our findings to the results in the literature.

Keywords: Common ordering extension, consistency, Szpirajn's lemma

JEL-classification: C60; D90; D63

1 Introduction

Szpilrajn's lemma (1930) states that every partial relation can be extended to a linear order. This result has many important applications in economic theory[1] of which Richter's revealed preference characterization (Richter, 1966) and Svensson's result on the existence of equitable and efficient welfare orderings on the set of infinite utility streams (Svensson, 1980) are especially noteworthy.

The literature on binary extensions has taken several directions and contains many interesting results of which we are only able to mention a few. Dushnik and Miller (1941) generalize Szpilrajn's lemma by showing that any strict partial order is equal to its ordering extensions. Building on the original result of Szpilrajn (1930), Suzumura (1976) shows that a relation has an ordering extension if and only if it is 'consistent', i.e. its transitive closure does not conflict with its asymmetric part. Donaldson and Weymark (1998) extends the result of Dushnik and Miller (1941) by showing that any quasiordering can be written as the intersection of its ordering extensions (see also Bossert (1999) for a proof that explicitly uses the result of Dushnik and Miller (1941)). Duggan (1999) provides a general extension result from which all previous mentioned results can be obtained as special cases. In an interesting paper, Banerjee and Pattanaik (1996) demonstrate that the maximal set of a quasi-ordering can be recovered by taking the union of the greatest sets of its ordering extensions (see also Suzumura and Xu (2003) for similar results). Many other papers in the literature deal with the characterization of the set of binary relations which have an ordering extension that satisfies some additional conditions. See, among others, Scapparone (1999) for the additional condition of convexity, Demuynck (2009) for the conditions of convexity, monotonicity and homotheticity, Jaffray (1975) and Bossert et al. (2002) for the condition of semicontinuity, Herden and Pallack (2002) for the condition of continuity and Clark (1993) and Demuynck and Lauwers (2009) for the condition of linearity. Finally we mention two researches that tackle the problem of extending a binary relation conditional

[1]We refer to Andrikopoulos (2009) for an thourough overview of the influence of Szpilrajn's lemma in economic theory

on an existing list of comparisons. Suzumura (2004) generalizes a result of Arrow (1963) by providing sufficient conditions for the existence of an ordering extension that coincides with a specified ordering on a certain subset of its domain. Alcantud (2009) tackles the problem of extending a quasi-ordering conditional on a finite list of ex-ante feasible comparisons. These two papers will be discussed in more depth in section 3.

In this research we study the issue concerning the existence of a common ordering extension. In a first part we consider two, not necessarily finite, collections, $\mathscr{C}_1$ and $\mathscr{C}_2$ of binary relations and we are concerned with the existence of an ordering, $\tilde{R}$, such that $\tilde{R}$ extends every relation in $\mathscr{C}_1$ and contains every relation in $\mathscr{C}_2$. Our main result characterizes the set of pairs $(\mathscr{C}_1, \mathscr{C}_2)$ for which such an ordering exists. From this result, we derive Suzumura's (1976) characterization, concerning the existence of an ordering extension for a single relation, as a corollary. We also extend the result of Donaldson and Weymark (1998) by characterizing the quasi-ordering that is obtained by taking the intersection over all these ordering extensions. We show the usefullness of these results by providing several examples.

In a second part of this paper, we restrict the setting to the special case where $\mathscr{C}_2$ is empty and $\mathscr{C}_1$ contains only two relations. This is also the setting of the papers by Suzumura (2004) and Alcantud (2009) and we compare our findings with their results. Our main result shows that, in this special setting, the necessary and sufficient conditions can be reformulated to include solely binary relations that are defined on a certain subset of the universal domain of alternatives.

Section 2 provides notation, definitions and the statements of the main results. We also include several examples. Section 3 discusses the special case where $\mathscr{C}_2$ is empty and $\mathscr{C}_1$ contains only two relations. All proofs are in the appendix.

2 Common ordering extensions

Let X be a non-empty set of alternatives. A *binary relation* R is a subset of the cartesian product $X \times X$. The *inverse* R^{-1} of R is the relation $\{(x,y) \in X \times X | (y,x) \in R\}$. The *asymmetric part*, $P(R)$ of R is given by $R - R^{-1}$ and its *symmetric part*, $I(R)$ is given by $R \cap R^{-1}$. We denote by Δ_X the *diagonal* relation on X, $\Delta_X = \{(x,x) \in X \times X | x \in X\}$.

A relation, R, is *reflexive* if $\Delta_X \subseteq R$. It is *transitive* if for all x, y and $z \in X$, $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$. The relation, R, is *complete* if for all $x, y \in X$, $(x,y) \in R$ or $(y,x) \in R$ or both. A complete relation is always reflexive. A complete and transitive relation is called an *ordering* while a reflexive and transitive relation is a *quasi-ordering*. The *transitive closure* $T(R)$ of a relation R is the set of pairs (x,y) such that there exists a finite number $n \in \mathbb{N}_0$ and a sequence $x = x_1, x_2, \ldots, x_n = y$ such that for all $i < n$, $(x_i, x_{i+1}) \in R$. It is easy to see —by induction— that a binary relation R is transitive if and only if $R = T(R)$. The transitive closure is *increasing:* $R \subseteq T(R)$, *idempotent:* $T(T(R)) = T(R)$ and *monotone:* if $R \subseteq Q$ then $T(R) \subseteq T(Q)$.

A relation $\tilde{R}$ is an *extension* of the relation R if $R \subseteq \tilde{R}$ and $P(R) \subseteq P(\tilde{R})$. A binary relation, R is *consistent* if for all $(x,y) \in T(R)$, $(y,x) \notin P(R)$. This property can be written more succinctly by the condition, $T(R) \cap P(R)^{-1} = \emptyset$. The following theorem is due to Suzumura

(1976).

Theorem 1 (Suzumura, 1976). *A binary relation has an ordering extension if and only if it is consistent.*

The proof of theorem 1 depends on Szpilrajn's lemma (Szpilrajn, 1930) which states that every partial order has a linear extension. The original proof of Szpilrajn uses Zermelo's well-ordering theorem which is equivalent to the Axiom of Choice and Zorn's lemma[2]. As such, theorem 1 contains a non-constructive argument. This implies that, although it is always possible to know if there exists an ordering extension, it is not always possible to actually construct such an extension. It is important to know that —as is usual in the literature— this non-constructive nature carries over to all other results in this paper.

Before we state our main result, we need one more definition. Consider two index sets A_1 and A_2 and let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two collections of binary relations on X:

$$\mathscr{C}_1 = \{R_\alpha \subseteq X \times X | \alpha \in A_1\} \qquad \text{and} \qquad \mathscr{C}_2 = \{R_\beta \subseteq X \times X | \beta \in A_2\}.$$

Observe that the sets $\mathscr{C}_1$ and $\mathscr{C}_2$ are not restricted to be finite or countable. We abuse previous terminology and say that the relation $\tilde{R}$ *extends* the collection $\mathscr{C}_1$ if $\tilde{R}$ extends all relations in $\mathscr{C}_1$ and we say that $\tilde{R}$ extends the ordered pair $(\mathscr{C}_1, \mathscr{C}_2)$ if $\tilde{R}$ extends $\mathscr{C}_1$ and for all $R_\beta \in \mathscr{C}_2$, $R_\beta \subseteq \tilde{R}$. In practice it will always be clear from the context if a given relation extends a single relation, R, a collection of relations, $\mathscr{C}_1$, or a pair of collections $(\mathscr{C}_1, \mathscr{C}_2)$. We are now ready to give our main result.

Theorem 2. *The pair $(\mathscr{C}_1, \mathscr{C}_2)$ has an ordering extension if and only if for all $\alpha \in A_1$:*

$$T\left(\bigcup_{\gamma \in A_1 \cup A_2} R_\gamma\right) \cap P(R_\alpha)^{-1} = \emptyset.$$

If we set $\mathscr{C}_2 = \emptyset$ and if we restrict $\mathscr{C}_1$ to contain a single relation R, then theorem 2 reproduces theorem 1. Hence, we obtain Suzumura's characterization as a corollary.

Consider a pair $(\mathscr{C}_1, \mathscr{C}_2)$ that satisfies the condition of theorem 2 and let Σ be the non-empty set of its ordering extensions. The following theorem generalizes the result of Donaldson and Weymark (1998) that every quasi-ordering equals the intersection of its ordering extensions.

Theorem 3. *Let $(\mathscr{C}_1, \mathscr{C}_2)$ satisfy the condition in theorem 2 and let Σ be the non-empty set of its ordering extensions, then:*

$$\Delta_X \cup T\left(\bigcup_{\gamma \in A_1 \cup A_2} R_\gamma\right) = \bigcap_{\tilde{R} \in \Sigma} \tilde{R}.$$

[2]In fact, Szpilrajn's lemma can be proved using the weaker condition that there exists a free ultrafilter.

We end this section by giving some examples.

Example 1: Let us start with a simple example. Let $X = \mathbb{R}^n_+$ be the set of consumption bundles of n goods. Consider an individual with an unknown preference ordering over X that extends a known relation V. Further, we assume that preferences are monotonic, i.e. if $x \geq y$, then x is at least as good as y. In other words, the true preference relation contains the relation $Q = \{(x, y)|x \geq y\}$. Then we can set $\mathscr{C}_1 = \{V\}$ and $\mathscr{C}_2 = \{Q\}$ and we know, from theorem 2 that there exists a monotonic ordering on X which extends V if and only if:

$$T(V \cup Q) \cap P(V)^{-1} = \emptyset.$$

Furthermore, theorem 3 states that the true preference relation contains the relation $\Delta_X \cup T(V \cup Q)$ and that this is the largest relation for which we are sure that this inclusion holds.

Now, assume that we know that the preference relation is not only monotonic, but also strict monotonic, i.e. if $x \geq y$ and $x \neq y$ then x is preferred to y. This agrees with the condition that the true preferen relation extends both V and Q. In this case, we set $\mathscr{C}_1 = \{V, Q\}$ and $\mathscr{C}_2 = \emptyset$ and we derive the following additional restriction.

$$T(V \cup Q) \cap P(Q)^{-1} = \emptyset.$$

Example 2: Let A be a subset of $\mathbb{R}$ of cardinality at least 2 and let $\geq$ be the usual greater then or equal ordering on $\mathbb{R}$. Consider the product space $X = \prod_{i \in \mathbb{N}} A$. If we interpret the elements of A as utility levels, then X represents the set of all infinite utility streams. The ith component, x_i, of x represents the utility level of generation i in state x. Let Q_i be the relation on X defined by,

$$(x, y) \in Q_i \text{ if } x_i \geq y_i \text{ and for all } j \neq i : x_j = y_j.$$

In words, $(x, y) \in Q_i$ if generation i has at least as much utility in state y as in state x while all other generations are indifferent. We say that a social welfare ordering $\tilde{R}$ on X satisfies the *Finite Pareto* principle if $\tilde{R}$ extends Q_i for all $i \in \mathbb{N}$.

Further, consider the relation V on X defined by $(x, y) \in V$ if for all $i \in \mathbb{N}$, $x_i > y_i$. We say that the social welfare ordering $\tilde{R}$ on X satisfies the *Weak Pareto* principle, if $\tilde{R}$ extends V.

It would seem that the notions of Finite Pareto and Weak Pareto are somehow related. However, this is not the case. In particular we show that there exists a social welfare ordering that satisfies the Finite Pareto principle and violates the Weak Pareto principle everywhere. In other words, there exists a social welfare ordering $\tilde{R}$ on X that extends Q_i for every $i \in \mathbb{N}$ and $\tilde{R}$ also extends V^{-1}.

Consider the collection $\mathscr{C}_1 = \{V^{-1}, Q_1, \ldots, Q_n, \ldots\}$. We need to show that this collection has an ordering extension. The relation V is asymmetric[3], hence, by theorem 2 it suffices to show that for $R = V^{-1} \cup \left(\bigcup_{i \in \mathbb{N}} Q_i\right)$:

$$T(R) \cap V = \emptyset$$

[3]A relation, V, is asymmetric if $(x, y) \in V$ implies that $(y, x) \notin V$.

and for all $i \in \mathbb{N}$

$$T(R) \cap P(Q_i)^{-1} = \emptyset.$$

Step 1. For two elements $x, y \in X$, let $A(x,y) = \{i \in \mathbb{N} | x_i > y_i\}$. It is easy to see that $A(x,y) \cap A(y,x) = \emptyset$ and $A(x,y) \cup A(y,x) \subseteq \mathbb{N}$.

We show that $(x,y) \in T(R)$ if and only if $A(x,y)$ has finite cardinality and $A(y,x)$ is empty or has infinite cardinality. We work by induction on the number n in the definition of the transitive closure. For $n = 1$, we immediately have that $|A(x,y)| \leq 1$ and that $A(y,x)$ either equals the empty set or the set $\mathbb{N}$. Let the conclusion hold for all positive integers $n < m$ and take the case where $n = m$. Then, we have that there are elements $x, z, y \in X$ such that $|A(x,z)|$ is finite and $|A(z,x)|$ is infinite or zero and $(z,y) \in R$. If $(z,y) \in Q_i$ for some $i \in \mathbb{N}$, then, $|A(x,y)|$ is equal to $|A(x,z)|$ plus one and if $(z,y) \in V^{-1}$, then $|A(x,y)|$ must be less than or equal to $|A(x,z)|$. Hence, if $|A(x,z)|$ is finite, then $|A(x,y)|$ must also be finite. If $(z,y) \in Q_i$ for some $i \in \mathbb{N}$ and $|A(z,x)|$ is infinite, then $|A(y,x)|$ is greater than or equal to $|A(z,x)|$ minus one, hence $|A(y,x)|$ is infinite. If $(z,y) \in Q_i$ for some $i \in \mathbb{N}$ and $|A(z,x)|$ is zero, then $|A(y,x)|$ is also zero. If $(z,y) \in V^{-1}$ and $|A(z,x)|$ is infinite, then $|A(y,x)|$ is greater than or equal to $|A(z,x)|$, hence, it is also infinite. Finally, we need to consider the case where $(z,y) \in V^{-1}$ and $|A(z,x)|$ is zero. As $|A(x,z)|$ is finite and $|A(z,x)|$ is zero, we see that the set $\{i \in \mathbb{N} | x_i = z_i\}$ has infinite cardinality. Further, we have that for all $i \in \mathbb{N}$, $y_i > z_i$. Conclude that $|A(y,x)|$ is infinite.

Step 2. We prove the claim by contradiction. Suppose, on the contrary, that $(x,y) \in T(R)$ and $(y,x) \in P(Q_i)$. Then $|A(y,x)| = 1$, a contradiction with the result of step 1. On the other hand, if $(x,y) \in T(R)$ and $(y,x) \in V^{-1}$, then $A(x,y) = \mathbb{N}$, again a contradiction with the result of step 1. Conclude that $\mathscr{C}_1$ has an ordering extension.

Example 3: Consider a set X and let let $\mathscr{C}_1$ be a collection of consistent relations in X such that for all $V, Q \in \mathscr{C}_1$ either $V \subseteq Q$ or $Q \subseteq V$. We show that the following conditions are equivalent:

- the collection $\mathscr{C}_1$ has an ordering extension.
- for all relations V and Q in $\mathscr{C}_1$, if $V \subseteq Q$, then Q is an extension of R.

In view of theorem 2, we need to show that the condition,

$$T\left(\bigcup_{V \in \mathscr{C}_1} V\right) \cap P(Q)^{-1} = \emptyset \text{ for all } Q \in \mathscr{C}_1,$$

is equivalent to the second condition. Let $R = \bigcup_{V \in \mathscr{C}_1} V$ and assume, on the contrary, that $(x,y) \in T(R)$ and $(y,x) \in P(Q)$ for some $Q \in \mathscr{C}_1$. Then there are elements $x = x_1, \ldots, x_n = y$ and relations $V_1, \ldots, V_{n-1} \in \mathscr{C}_1$ such that for all $i < n$, $(x_i, x_{i+1}) \in V_i$. Let V be the largest –with respect to set inclusion– relation in the set $\{V_1, \ldots, V_{n-1}, Q\}$. The relation V extends all relations in $\{V_1, \ldots, V_{n-1}, Q\}$, hence, $(x, y) \in T(V)$ and $(y, x) \in P(V)$. This contradicts the consistency of V.

On the other hand, if $V, Q \in \mathscr{C}_1$, $V \subseteq Q$ and, on the contrary, $(x, y) \in P(V)$ while $(y, x) \in Q$, we have that $(y, x) \in T(R)$ and $(x, y) \in P(V)$, a contradiction.

Notice that theorem 3 implies that the intersection of all the ordering extensions of the relations in $\mathscr{C}_1$ must equal the relation $\Delta_X \cup T\left(\bigcup_{V \in \mathscr{C}_1} V\right)$

Above result can be applied to various settings. We give one simple example. Consider – analogue to the setting in example 2– the set $X = \prod_{i \in \mathbb{N}} A$ with $A \subseteq \mathbb{R}$. For an element $x = (x_1, \ldots, x_n, \ldots) \in X$, let $x^n_{(i)}$ be the i-th smallest element in the set $\{x_1, \ldots, x_n\}$.

For all $n \in \mathbb{N}$, consider the the subset X_n of $X \times X$ given by the pairs (x, y) such that for all $i > n$, $x_i \geq y_i$. For each $n \in \mathbb{N}$, consider the relation R_n on X_n defined by $(x, y) \in R_n$ if and only if:

$$\sum_{i \leq k} x^n_{(i)} \geq \sum_{i \leq k} y^n_{(i)} \qquad \text{for all } k \leq n$$

It is easy to see that all relations R_n are transitive, hence they are also consistent. In order to apply previous result it suffices to show that for all numbers $n, m \in \mathbb{N}$ with $n \leq m$, R_m extends R_n. We refer to Bossert et al. (2007, step 1 in the proof of theorem 1) for a proof of this assertion and for a discussion and intuition of this —and similar— ordering extension results in the literature related to the ordering of infinite utility streams.

3 A special case

In this section, we restrict our setting by assuming that $\mathscr{C}_2 = \emptyset$ and that $\mathscr{C}_1$ only contains two relations V and Q. These restrictions allow us to compare our results with those obtained by Suzumura (2004) and Alcantud (2009).

Consider the binary operator $\circ$ on $X \times X$ which is defined by the condition that for two relations R and S; $(x, y) \in R \circ S$ if and only if there exists an element $z \in X$ such that $(x, z) \in R$ and $(z, y) \in S$. Further, denote by $TI(R)$ the relation $T(R) \cup \Delta_X$. The relation $TI(R)$ is the smallest —with respect to set inclusion— quasi-ordering containing R. For any relation R on X, define two subsets of X:

$$D_R = \{x \in X \mid \exists y \in X, (x, y) \in R\},$$

$$I_R = \{y \in X \mid \exists x \in X, (x, y) \in R\}.$$

Call D_R the *domain* of R in X and call I_R the *image* of R in X. From now on, fix the relation Q. Then for a relation S on X denote by $\underline{S}$ the restricted relation $S \cap (D_Q \times D_Q)$ and denote by $\overline{S}$ the restricted relation $S \cap (I_Q \times I_Q)$. If the set I_Q or D_Q only contains a fraction of the elements in X it would be beneficial to rewrite our conditions for the existence of an ordering extension in terms of relations that are restricted to these domains, as in following theorem.

Theorem 4. *Let $TI(V) = R$. Then the following conditions are equivalent:*

i.) $\mathscr{C}_1 = \{V, Q\}$ *has an ordering extension,*

ii.) $TI\left(\underline{Q \circ R}\right) \cap \left(\underline{P(Q) \circ R}\right)^{-1} = \emptyset$ *and* $TI\left(\underline{Q \circ R}\right) \cap \left(\underline{Q \circ P(R)}\right)^{-1} = \emptyset$ *and* V *is consistent,*

iii.) $TI\left(\overline{R \circ Q}\right) \cap \left(\overline{P(R) \circ Q}\right)^{-1} = \emptyset$ *and* $TI\left(\overline{R \circ Q}\right) \cap \left(\overline{R \circ P(Q)}\right)^{-1} = \emptyset$ *and* V *is consistent.*

Of course, it is possible to switch the roles of V and Q in the result of theorem 4. So, for practical purposes, it is of interest to decide which relation fullfills which role. The following two guidelines may be usefull:

- If one of the two relations is a quasi-ordering, it is probably a good idea to identify V with this relation. This would simplify the conditions of theorem 4 in two ways. First, any quasi-ordering is consistent, such that the consistency condition of (ii) and (iii) is always satisfied. Second, for a quasi-ordering, one has the identity $V = TI(V) = R$ such that we can replace R by V in both conditions (ii) and (iii).
- if one of the two relations has a very small domain or image, it may be best to let this relation take the place of Q.

The smallest consistent relation in X is the empty relation. Using $V = \emptyset$ and, consequently, $R = \Delta_X$ in theorem 4 gives the following corollary.

Corollary 1. *The following conditions are equivalent:*

- *the relation* Q *has an ordering extension,*
- $T(Q) \cap P(Q)^{-1} = \emptyset$,
- $T\left(\underline{Q}\right) \cap P\left(\underline{Q}\right)^{-1} = \emptyset$,
- $T\left(\overline{Q}\right) \cap P\left(\overline{Q}\right)^{-1} = \emptyset$.

Next, let us discuss the two closely related papers mentioned in the beginning of this section. Alcantud (2009) focusses on the case where the relation Q is finite and V is a quasi-ordering. He has two extension result. For his first result he considers a finite list of elements $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ and assumes that $Q = \{(b_i, a_i) | i = 1, \ldots, n\}$ is an asymmetric relation. If we apply theorem 4 to this setting, we see that $I_Q = \{a_1, \ldots, a_n\}$. Hence, we can construct the relation $S = \overline{V \circ Q}$ given by $(a_i, a_j) \in S$ if and only if $(a_i, b_j) \in V$. Theorem 4 lets us conclude that $\mathscr{C}_1 = \{V, Q\}$ has an ordering extension if and only if: $T(S) \cap S^{-1} = \emptyset$, or in other words, if S is acyclic (see Alcantud, 2009, theorem 1). His second result considers the case where $Q = \{(b_i, a_i) | i = 1, \ldots, n\} \cup \{(a_j, b_j) | j = l + 1, \ldots, n\}$. The characterization derived by Alcantud (2009) for this case is very elegant but quite intricate and it would take too much space to translate his extension result to our framework without providing interesting new insights. We refer to the paper of Alcantud (2009) for some interesting applications of his results to economic theory.

Suzumura's main theorem (Suzumura, 2004) states the following.

Theorem 5 (Suzumura, 2004)**.** *Let V be a binary relation on X and let A be a subset of X such that, if $x \neq y$ and $x, y \in A$, then $(x, y) \neq T(V)$. Let Q be an ordering on A. Then there exists an ordering extension R of V such that the restriction of R on A coincides with Q if and only if V is consistent.*

In order to see that this theorem follows from theorem 4 observe that consistency of V is indeed necessary to have an ordering extension. Let us show that it is also sufficient. The domain and image of Q is equal to A. Set $R = TI(V)$. The premises of theorem 4 imply that $\underline{Q \circ R} = Q$, $\overline{P(Q) \circ R} = P(Q)$ and $\underline{Q \circ P(R)} = \emptyset$. The second condition of theorem 4 is satisfied if $TI\left(\underline{Q \circ R}\right) \cap \left(\overline{P(Q) \circ R}\right)^{-1} = \emptyset$, which is equivalent to the condition that $T(Q) \cap P(Q)^{-1} = \emptyset$. This condition is always satisfied —because Q equals its transitive closure— and, hence, $\mathscr{C}_1$ has an ordering extension. Suzumura provides several interesting applications of this theorem to the field of welfare economics.

In order to illustrate the usefulness of theorem 4 we conclude this section with an example that cannot be solved using the results of Suzumura (2004) or Alcantud (2009).

Example 4: Let X be a subset of the set of consumption vectors $\mathbb{R}^n$ and assume that we know that the preference relation of the individual extends a certain relation V. Assume also that there is a bundle $a = \{a_1, \ldots, a_n\}$ below which the individual cannot survive. Let $Q = \{(x, y) |$ for some $i, x_i \geq a_i$ and for all $i, y_i < a_i\}$. It seems natural to require that the true preference relation extends the relation Q. Also observe that the image of Q equals the set $I_Q = \{x \in X |$for all $i, x_i < a_i\}$.

Let $S = \overline{R \circ Q}$ with $R = TI(V)$. The relation S contains all elements (x, z) from $I_Q \times I_Q$ such that there exists an $y \in X$ for which $(x, y) \in R$ and $(y, z) \in Q$. It is easy to see that theorem 4 implies that the following conditions are equivalent:

- The set $\mathscr{C}_1 = \{V, Q\}$ has an ordering extension,
- $T(S) \cap S^{-1} = \emptyset$ and V is consistent.

The advantage of using the relation S instead of $V \cup Q$ is that —if a is not to large– the image of S contain only a small subset of X.

4 Appendix

Proof of Theorem 2: To simplify notation, we define the relation $R = \bigcup_{\gamma \in A_1 \cup A_2} R_\gamma$.

($\leftarrow$) Obviously, $T(R)$ is a transitive relation, hence, it is consistent. By Theorem 1, $T(R)$ has an ordering extension, say $\tilde{R}$. Clearly, $R_\beta \subseteq \tilde{R}$ for all $\beta \in A_2$ so we only need to verify that $\tilde{R}$ extends R_α for all $\alpha \in A_1$. Observe that $R_\alpha \subseteq \tilde{R}$. Now, assume on the contrary that there exists an $\alpha \in A_1$ such that $(x, y) \in P(R_\alpha)$ and $(y, x) \in \tilde{R}$. From this, it follows that $(x, y) \in I(\tilde{R})$. Also, $(y, x) \in T(R)$. Indeed, if on the contrary $(x, y) \in P(T(R))$, then also

$(x,y)\in P(\tilde{R})$, contradicting the previous result that $(y,x)\in I(\tilde{R})$. Conclude that $(y,x)\in T(R)$ and $(x,y)\in P(R_\alpha)$. This, however, contradicts the condition in the theorem.

($\rightarrow$) On the other hand, let $\tilde{R}$ be an ordering extension of $(\mathscr{C}_1,\mathscr{C}_2)$. Obviously, $R\subseteq\tilde{R}$ and, from monotonicity of the transitive closure, $T(R)\subseteq T(\tilde{R})=\tilde{R}$. Further, for all $\alpha\in A_1$; $P(R_\alpha)\subseteq P(\tilde{R})$. Conclude that the condition in the theorem, $T(R)\cap P(R_\alpha)^{-1}=\emptyset$, holds for all $\alpha\in A_1$.

Proof of Theorem 3: Again, we define the relation $R=\bigcup_{\gamma\in A_1\cup A_2}R_\gamma$.

($\subseteq$) Let $(x,y)\in T(R)\cup\Delta_X$. Clearly $R\subseteq\tilde{R}$ for all $\tilde{R}\in\Sigma$ and every relation in Σ is reflexive. Hence, by monotonicity of the transitive closure $(x,y)\in T(\tilde{R})=\tilde{R}$ for all $\tilde{R}\in\Sigma$. Conclude that $(x,y)\in\bigcap_{\tilde{R}\in\Sigma}\tilde{R}$.

($\supseteq$) To see the reverse, let $(x,y)\in\bigcap_{\tilde{R}\in\Sigma}\tilde{R}$ and assume, on the contrary, that $(x,y)\notin T(R)\cup\Delta_X$. If $(y,x)\in P(R_\alpha)$ for some $\alpha\in A_1$, then $(y,x)\in P(R)$ for all $R\in\Sigma$, which is impossible. Hence (x,y) and $(y,x)\notin R_\alpha$ for all $\alpha\in A_1$ and $(x,y)\notin R_\beta$ for all $\beta\in A_2$.

Consider the binary relation $V=\{(y,x)\}$ and define $\mathscr{C}_1'=\mathscr{C}_1\cup\{V\}$ and $R'=R\cup V$. Let us show that the pair $(\mathscr{C}_1',\mathscr{C}_2)$ satisfies theorem 2.

Assume, on the contrary, that there is a pair $(a,b)\in T(R')$ such that $(b,a)\in P(R_\alpha)$ for some $\alpha\in A_1$ or that $(b,a)=(y,x)$. If $(a,b)\in T(R)$ then evidently, $(b,a)\notin P(R_\alpha)$ for all $\alpha\in A_1$ and by assumption $(b,a)\neq(y,x)$. Therefore, it must be that $(a,b)\in T(R')-T(R)$ and $(b,a)\in P(R_\alpha)$ for some $\alpha\in A_1$ or $(b,a)=(y,x)$.

If the first is the case, we know that there must be a finite number of elements $a=x_1,\dots,x_n=b$ in X such that for all $i<n$: $(x_i,x_{i+1})\in R$ or $(x_i,x_{i+1})=(y,x)$. Let x_l be the last element in this sequence equal to x and let x_f be the first element in this sequence equal to y. We know that x_l and x_f exists because $(a,b)\notin T(R)$. It follows that $(x_l,x_f)=(x,y)\in T(R)$, a contradiction.

If the second is the case, i.e. $(b,a)=(y,x)$, define x_l to be the last instance of x in the sequence and we derive that $(x_l,x_n)=(x,y)\in T(R)$, again a contradiction.

Conclude that $(\mathscr{C}_1',\mathscr{C}_2)$ satisfies the condition of theorem 2. As such, it has an ordering extension, say $\tilde{R}$. This relation is also an extension of $(\mathscr{C}_1,\mathscr{C}_2)$, hence, it is in Σ. Further, $\tilde{R}$ extends the relation $V=\{(y,x)\}$, hence $(y,x)\in P(\tilde{R})$, a contradiction.

Proof of Theorem 4 (i)$\rightarrow$(ii) and (i)$\rightarrow$(iii) are obvious in the light of theorems 1 and 2. We will focus on the case (ii)$\rightarrow$(i) noticing that (iii)$\rightarrow$(i) is very similar. We split the proof in a number of different steps.

Step 1: *If V is consistent, then $\tilde{R}$ is an ordering extension of $\mathscr{C}_1=\{V,Q\}$ if and only if $\tilde{R}$ is an ordering extension of $\mathscr{C}_1'=\{R,Q\}$, with $R=TI(V)$.*

Assume that V is consistent and let $\mathscr{C}_1'$ have an ordering extension, $\tilde{R}$. Then R extends V and $\tilde{R}$ extends R, hence $\tilde{R}$ extends V. Conclude that $\tilde{R}$ extends $\mathscr{C}_1$.

To see the reverse, let $\mathscr{C}_1$ have an ordering extension $\tilde{R}$. Then $R \subseteq \tilde{R}$ because $\tilde{R}$ is reflexive, transitive and it contains V. If $(x,y) \in P(R)$, then $(x,y) \in TI(V)$. Hence, there is a number n such that $x = x_1, \ldots, x_n = y$ and for all $i < n$, $(x_i, x_{i+1}) \in V$. Clearly, there must be a number $j < n$ such that $(x_j, x_{j+1}) \in P(V)$, otherwise, $(y,x) \in T(V) \subseteq R$. Now, assume on the contrary, that $(y,x) \in I(\tilde{R})$. From $V \subseteq \tilde{R}$ and transitivity of $\tilde{R}$, we see that $(x_{j+1}, y) \in \tilde{R}$, $(y,x) \in \tilde{R}$ and $(x, x_j) \in \tilde{R}$. Conclude that $(x_{j+1}, x_j) \in \tilde{R}$ and $(x_j, x_{j+1}) \in P(V)$, a contradiction.

Step 2: If R is a quasi-ordering, then

$$T(R \cup Q) = R \cup \left[R \circ TI\left(\underline{Q \circ R}\right) \circ Q \circ R \right].$$

($\subseteq$) Let $S = R \circ TI\left(\underline{Q \circ R}\right) \circ Q \circ R$ and consider $(x,y) \in T(R \cup Q)$. Hence, there is a number n and a sequence $x = x_1, \ldots, x_n = y$ such that for all $i < n$, $(x_i, x_{i+1}) \in R \cup Q$. We proof the step by induction, showing that it holds for all (x, x_i), $i \leq n$. Consider the case $i = 2$. Then $(x, x_2) \in R \cup Q$. If $(x, x_2) \in R$, nothing has to be proved. If $(x, x_2) \in Q$, we let $(x,x) \in R$, $(x,x) \in TI\left(\underline{Q \circ R}\right)$, $(x, x_2) \in Q$ and $(x_2, x_2) \in R$. Hence $(x,y) \in S$. Assume that $(x, x_i) \in R \cup S$ for all $i \leq m$ and consider the case with $i = m+1$. Then $(x, x_m) \in R \cup S$ and $(x_m, x_{m+1}) \in R \cup Q$. If $(x, x_m) \in R$ and $(x_m, x_{m+1}) \in R$ it follows, by transitivity of R that $(x, x_{m+1}) \in R$. If $(x, x_m) \in R$ and $(x_m, x_{m+1}) \in Q$, then $(x, x_m) \in R$, $(x_m, x_m) \in TI\left(\underline{Q \circ R}\right)$, $(x_m, x_{m+1}) \in Q$ and $(x_{m+1}, x_{m+1}) \in R$, hence $(x, x_{m+1}) \in S$. If $(x, x_m) \in S$ and $(x_m, x_{m+1}) \in R$, then $(x, x_{m+1}) \in \left[R \circ TI\left(\underline{Q \circ R}\right) \circ Q \circ R \right] \circ R = S$ by transitivity of R. If $(x, x_m) \in S$ and $(x_m, x_{m+1}) \in Q$, then $(x, x_{m+1}) \in \left[R \circ TI\left(\underline{Q \circ R}\right) \circ Q \circ R \right] \circ Q$ $= \left[R \circ TI\left(\underline{Q \circ R}\right) \circ \underline{Q \circ R} \circ Q \right] = \left[R \circ TI\left(\underline{Q \circ R}\right) \circ Q \right] \subseteq \left[R \circ TI\left(\underline{Q \circ R}\right) \circ Q \circ R \right] = S$. Therefore the argument is valid for $i = m+1$. Using a simple induction argument it follows that $(x,y) \in R \cup S$.

($\supseteq$) To see the reverse, notice that $R \subseteq T(R \cup Q)$, $TI\left(\underline{Q \circ R}\right) \subseteq T(R \cup Q)$ and $Q \subseteq T(R \cup Q)$. Hence, $R \cup S \subseteq T(R \cup Q)$.

Step 3: The following rules apply for all relations Q and V:

$$\begin{gathered} Q \cap V^{-1} = \emptyset \leftrightarrow [Q \circ V] \cap \Delta_X = \emptyset \\ [Q \circ V] \cap \Delta_X = \emptyset \leftrightarrow [V \circ Q] \cap \Delta_X = \emptyset \end{gathered} \tag{1}$$

The proof is straightforward, so we leave it to the reader.

Step 4: We complete the proof. By step 1, $\mathscr{C}_1 = \{V, Q\}$ has an ordering extension if and only if $\mathscr{C}_1' = \{R, Q\}$ has an ordering extension where $R = TI(V)$. By theorem 2, this is equivalent to:

$$T(R \cup Q) \cap P(R)^{-1} = \emptyset \text{ and } T(R \cup Q) \cap P(Q)^{-1} = \emptyset.$$

Let us show its equivalence with condition (ii). Start with the first condition:

$$
\begin{array}{lr}
T(R \cup Q) \cap P(R)^{-1} = \emptyset, & \\
\leftrightarrow \left(R \cup \left[R \circ TI\left(\underline{Q \circ R}\right) \circ Q \circ R\right]\right) \cap P(R)^{-1} = \emptyset, & \text{(by step 2)} \\
\leftrightarrow \left[R \circ TI\left(\underline{Q \circ R}\right) \circ Q \circ R \circ P(R)\right] \cap \Delta_X = \emptyset & \text{(by step 3)} \\
\qquad \text{and} \qquad R \cap P(R)^{-1} = \emptyset & \\
\leftrightarrow \left[TI\left(\underline{Q \circ R}\right) \circ Q \circ R \circ P(R) \circ R\right] \cap \Delta_X = \emptyset & \text{(by step 3 and consistency of } R\text{)} \\
\leftrightarrow \left[TI\left(\underline{Q \circ R}\right) \circ \underline{Q \circ P(R)}\right] \cap \Delta_X = \emptyset & \text{(by transitivity of } R\text{)} \\
\leftrightarrow TI\left(\underline{Q \circ R}\right) \cap \left(\underline{Q \circ P(R)}\right)^{-1} = \emptyset & \text{(by step 3)}
\end{array}
$$

Now, for the second condition:

$$
\begin{array}{lr}
T(R \cup Q) \cap P(Q)^{-1} = \emptyset, & \\
\leftrightarrow \left(R \cup \left[R \circ TI\left(\underline{Q \circ R}\right) \circ Q \circ R\right]\right) \cap P(Q)^{-1} = \emptyset, & \text{(by step 2)} \\
\leftrightarrow \left[R \circ TI\left(\underline{Q \circ R}\right) \circ Q \circ R \circ P(Q)\right] \cap \Delta_X = \emptyset & \text{(by step 3)} \\
\qquad \text{and} \qquad [R \circ P(Q)] \cap \Delta_X = \emptyset & \text{(by step 3)} \\
\leftrightarrow \left[TI\left(\underline{Q \circ R}\right) \circ \underline{Q \circ R} \circ P(Q) \circ R\right] \cap \Delta_X = \emptyset & \text{(by step 3)} \\
\qquad \text{and} \qquad [P(Q) \circ R] \cap \Delta_X = \emptyset & \text{(by step 3)} \\
\leftrightarrow \left[TI\left(\underline{Q \circ R}\right) \circ \underline{P(Q) \circ R}\right] \cap \Delta_X = \emptyset & \\
\qquad \text{and} \qquad \left[\underline{P(Q) \circ R}\right] \cap \Delta_X = \emptyset & \\
\leftrightarrow TI\left(\underline{Q \circ R}\right) \cap \left(\underline{P(Q) \circ R}\right)^{-1} = \emptyset & \text{(by step 3)}
\end{array}
$$

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