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WORKING PAPER

Structural Mobility, Exchange Mobility and Subgroup Consistent Mobility Measurement – US–German Mobility Measurements Revisited

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Structural Mobility, Exchange Mobility, and Subgroup Consistent Mobility Measurement - US-German Mobility Rankings Revisited *

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Abstract

We formalize the concept of structural mobility and use the framework of subgroup consistent mobility measurement to derive a relative and an absolute measure of mobility that is increasing both in upward structural mobility and exchange mobility. In our empirical illustration, we contribute substantively to the ongoing debate about mobility rankings between the USA and Germany.

Keywords: structural mobility, exchange mobility, income mobility, subgroup consistency.

JEL classification: D31, D63

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1 Introduction

Exchange and structural mobility are old and recurrent concerns in economics and sociology. Informally, exchange mobility is said to increase if the correlation between incomes in two successive periods decreases while keeping the marginal income distributions constant. By contrast, structural mobility compares situations that have different marginal distributions in the two periods. It amounts to factual mobility "caused by differential change in the stratum distribution" (Yasuda (1964), p.16) or -in an intergenerational context- by "the amount of mobility generated by the fact that the distribution among social strata experienced by the sons differs from the corresponding experience of their fathers" (Boudon (1973), p.17). The formal literature has focused on exchange mobility, whereas the concept of structural mobility has received only little attention.

However, students' verbal responses in questionnaire studies generally agree that structural mobility matters in both the measurement and evaluation of mobility (Bernasconi and Dardanoni (2005)). Moreover, to the best of our knowledge, in the axiomatic literature structural mobility has never been analyzed. We provide a formal definition of the concept, treating it as a directional concept such that more upward (downward) structural mobility increases (decreases) the index and show that it can be reconciled with exchange mobility.

We conduct our analysis within the framework of replication invariant, subgroup consistent mobility measurement. This class has a strong intuitive appeal as the following motivating example illustrates. Suppose we manage to identify a group in the population with a low income mobility (e.g. in an intergenerational context children from blue collar workers). If we manage to increase this group's upward mobility while keeping the mobility of other groups constant, then subgroup consistency ensures that total mobility in society has increased.[1] Subgroup consistency has long been accepted in inequality and poverty analyses as an important property; it is only recently that researchers have considered and formulated subgroup consistent mobility measures. Moreover, Schluter and Trede (2003) have highlighted the importance of transparent aggregation of income changes, and subgroup consistency ensures such transparency. We therefore consider the implications of exchange mobility and structural mobility for the class of subgroup consistent mobility indices, especially since existing measures do not comply with the notion of structural mobility. We characterize simple replication invariant subgroup consistent relative and absolute mobility indices that are increasing in exchange mobility and increasing (decreasing) in more upward (downward) structural mobility.

We demonstrated the usefulness of our framework by revisiting the ongoing debate about income mobility comparisons between the USA and Germany (see e.g. Burkhauser et al, 1997a and 1997b, Maasoumi and Trede, 2001, Gottschalk and Spolaore, 2002, and Schluter and Trede, 2003). The point of departure of this debate is the observation that when using standard mobility measures Germany is ranked, contrary to received wisdom, more mobile than the US. We contribute substantively to this debate by (i) showing that the standard mobility measures are inconsistent with our notions of upward structural and exchange mobility, and (ii) demonstrating that the US typically does exhibit more joint upward structural and exchange mobility than Germany.

The outline of this paper is as follows. In the next section we introduce the framework and core axioms and axiomatize the two notions of mobility. In Section 3 we derive a class

[1]Total mobility increases irrespective of how the change affects the relative positions of the remaining population. Foster and Sen (1997, p.156-163) discuss the pros and cons of this argument in the context of inequality measurement.

of mobility measures which are increasing in structural mobility and exchange mobility. Section 4 presents our empirical illustration about mobility rankings between the USA and Germany. All proofs are gathered in the Appendix.

2 Notation and Core Axioms

Let $y_1 \in \mathbb{R}^n_{++}$ be transformed through some dynamic process into $y_2 \in \mathbb{R}^n_{++}$; the vector (y_1, y_2) belongs to $D = \cup_{n=1}^{\infty} \mathbb{R}^{2n}_{++}$. In this paper we examine subgroup consistent mobility measures. To this end index the group by g, and, for notational simplicity, consider only two groups of individuals with $g \in \{1, 2\}$. Let $P = \{N^1, N^2\}$ be a partition of the set $N = \{1, \ldots, n\}$ in two non-overlapping subsets and let $\mathcal{P}$ denote the set of all such possible partitions of N. For each group g of size N^g, the income vectors are partitioned correspondingly into (y_1^g, y_2^g).

A *replication invariant subgroup consistent (RISC) mobility index* is a non-constant function $M : D \to \mathbb{R}$, continuous in its arguments, whose value indicates the amount of mobility in moving from a distribution y_1 to y_2, and which satisfies the following axioms:

RISC.1 [Anonymity] $M(y_1, y_2)$ is symmetric: $M(y_1, y_2) = M(y_1', y_2')$ whenever the vectors y_1' and y_2' are obtained after applying the same permutation on y_1 and y_2, respectively,

RISC.2 [Replication Invariance] $M(y_1, y_2) = M(y_1', y_2')$, whenever the vectors y_1' and y_2' are obtained after applying the same replication on y_1 and y_2, respectively,

RISC.3 [Subgroup consistency] For all $P \in \mathcal{P} : M(y_1^1, y_1^2, y_2^1, y_2^2) > M(y_1^{1\prime}, y_1^{2\prime}, y_2^{1\prime}, y_2^{2\prime})$ whenever $M(y_1^1, y_2^1) > M(y_1^{1\prime}, y_2^{1\prime})$ and $M(y_1^2, y_2^2) = M(y_1^{2\prime}, y_2^{2\prime})$.

Anonymity is generally accepted as a property for mobility measures. Replication invariance is also very common. The third requirement is subgroup consistency which has been considered in the literature on mobility measurement before (see, e.g., Fields and Ok (1999) and D'Agostino and Dardanoni (2007)).

We now formalize our two basic mobility properties. Our first mobility axiom concerns exchange mobility. Exchange mobility is a property of the joint distribution of incomes in both periods while the marginal distributions in both periods are kept fixed. It requires that if there is a positive association between the incomes in two periods for two pairs of incomes (y_{1i}, y_{2i}) and (y_{1j}, y_{2j}), then swapping y_{2i} and y_{2j} or swapping y_{1i} and y_{1j} increases mobility since it decreases the positive association between the income vectors of both periods. Given a vector $x = (x_1, \ldots, x_i, \ldots, x_j, \ldots x_n) \in \mathbb{R}^n_{++}$, define $x(\sigma_{ij}) = (x_1, \ldots, x_j, \ldots, x_i, \ldots x_n) \in \mathbb{R}^n_{++}$. The Exchange Mobility (EM) axiom states:

EM For all $i, j \in N$ and all $y_1, y_2, y_1(\sigma_{ij}), y_2(\sigma_{ij}) \in \mathbb{R}^n_{++} : (y_{1i} - y_{1j})(y_{2i} - y_{2j}) > 0 \Rightarrow M(y_1, y_2(\sigma_{ij})) > M(y_1, y_2)$ and $M(y_1(\sigma_{ij}), y_2) > M(y_1, y_2)$.

While the interpretation of EM[2] is clear and EM is often discussed in the literature (see, e.g., Atkinson (1981), Markandya (1982) and Dardanoni (1993)) its implications for a mobility index of the general form $M(y_1, y_2)$ are not. By imposing more structure on the

[2]Both swaps in the EM axiom decrease the joint probability distribution at (y_{1i}, y_{2i}) and (y_{1j}, y_{2j}) and increases it at (y_{1i}, y_{2j}) and (y_{1j}, y_{2i}). Thus, since the condition on the swaps is that $(y_{1i} - y_{1j})(y_{2i} - y_{2j}) > 0$, probability mass is pushed away from the diagonal. This is the equivalent, in the discrete case, of the mobility increasing transformation as introduced by Atkinson (1981).

mobility index using the RISC axioms we uncover the implications of EM for RISC mobility measures in Section 3.

Our second axiom concerns structural mobility and is concerned with changes in the marginal distributions. It consists out of two components. Given a vector $x = (x_1, \ldots, x_i, \ldots x_n) \in \mathbb{R}^n_{++}$, consider any $\varepsilon \in \mathbb{R}_{++}$ such that $x^{+\varepsilon(i)} = (x_1, \ldots, x_i + \varepsilon, \ldots x_n) \in \mathbb{R}^n_{++}$. Clearly the distribution $x^{+\varepsilon(i)}$ first order stochastically dominates the distribution x. (i) Assume that starting from y_1 a second period income distribution $y_2^{+\varepsilon(i)}$ is reached instead of y_2. The process that changes y_1 into the process $y_2^{+\varepsilon(i)}$ has more upward structural mobility than the process changing y_1 into y_2 since, starting from the same initial distribution of incomes (or social strata), a better distribution of income (with more desirable social strata) is reached. (ii) Assume now that a second period income distribution y_2 is reached from a first period income distribution $y_1^{+\varepsilon(i)}$ instead of from y_1. Here, the process that changes $y_1^{+\varepsilon(i)}$ into y_2 has less upward structural mobility than the process that changes y_1 into y_2 since the same second period income (or social strata) distribution is reached starting from a better distribution of income (or one with more attractive social strata). Formally this is captured by First Structural Mobility (FSM) axiom which combines the following properties:

FSM.1 For all $i \in N$ and all $y_1, y_2 \in \mathbb{R}^n_{++} : M\left(y_1, y_2^{+\varepsilon(i)}\right) > M(y_1, y_2)$.

FSM.2 For all $i \in N$ and all $y_1, y_2 \in \mathbb{R}^n_{++} : M(y_1, y_2) > M\left(y_1^{+\varepsilon(i)}, y_2\right)$.

(FSM.1) requires the index to be increasing in its last n arguments, while (FSM.2) requires it to be decreasing in its first n arguments. We note that it follows from (FSM.2) that the measurement of structural mobility contradicts the Pareto principle, interpreted to require that an ordering of income vectors is increasing in incomes of both periods. Any social welfare approach to the measurement of mobility that respects this Pareto principle[3] will never be able to capture the notion of structural mobility.

It is possible to look at a further structural mobility axiom that changes the marginals by changing the degree of inequality in the marginals (or the diversity of the social strata distributions) without affecting the covariance of the incomes in two periods. It is less clear a priori which kind of sensitivity to inequality in the marginal distributions is desirable from a structural mobility perspective. Fortunately these second structural mobility axioms play no role in our characterisation of mobility measures. However, as we feel that the sensitivity to the inequality in marginals is potentially important, we believe it worthwhile to verify the kind of sensitivity in this respect shown by our mobility measures.

Given a vector $x = (x_1, \ldots, x_i, \ldots, x_j, \ldots x_n) \in \mathbb{R}^n_{++}$, consider $\delta \in \mathbb{R}_{++}$ such that $x^{\delta(ij)} = (x_1, \ldots, x_i - \delta, \ldots, x_j + \delta, \ldots x_n) \in \mathbb{R}^n_{++}$, which is identical to x, except for a transfer δ that took place from i to j. (i) Assume that starting from y_1 a second period income distribution $y_2^{\delta(ij)}$ is reached that has more inequality than another distribution y_2 while the covariance between the distributions is not affected. In that case, it can be argued that the process that changes y_1 into $y_2^{\delta(ij)}$ is more mobile than the process changing y_1 into y_2 since starting from the same first period income (or social strata) distribution a distribution is reached with more dispersion (a greater diversity of social strata). (ii) Assume that a second period income distribution y_2 is reached from a first period income distribution $y_1^{\delta(ij)}$ that had more inequality than another first period income distribution y_1 while the

[3]This approach has been quite popular, see e.g. Atkinson (1981), Markandya (1982) or Chakravarty et al. (1985).

covariance between the distributions is not affected. Here it can be argued that the process changing $y_1^{\delta(ij)}$ into y_2 is less mobile than the process changing y_1 into y_2 since the same income (or social strata) distribution is reached starting with a more unequal distribution of income (or a distribution with a greater diversity of social strata). (iii) Assume that a second period income distribution $y_2^{\delta_2(ij)}$ is reached from a first period income distribution $y_1^{\delta_1(kl)}$ and both have more inequality than y_2 and y_1, respectively, while the covariance between the distributions is not affected. One might then claim that the process changing $y_1^{\delta_1(kl)}$ into $y_2^{\delta_2(ij)}$ is more mobile than the process changing y_1 into y_2 since the dynamic process occurs in societies with more income inequality (or more diverse social strata in both periods). Note that to ensure that the covariance between two distributions is not affected by a transfer from i to j in a marginal distribution, we have to impose that the marginal distribution that is not changed belongs to $R^n_{++}(ij)$, which, given $i, j \in N$, is defined as $\left\{x \in \mathbb{R}^n_{++} : x_i = x_j\right\}$. Keeping this in mind, we can define three versions of our Second Structural Mobility (SSM) Axiom:

SSM.1 For all $i, j \in N$ with $y_{2j} \geq y_{2i} > \delta \in \mathbb{R}_{++}$, all $y_1 \in R^n_{++}(ij)$ and all $y_2 \in \mathbb{R}^n_{++}$ $: M\left(y_1, y_2^{\delta(ij)}\right) > M\left(y_1, y_2\right).$

SSM.2 For all $i, j \in N$ with $y_{1j} \geq y_{1i} > \delta \in \mathbb{R}_{++}$, all $y_2 \in R^n_{++}(ij)$ and all $y_1 \in \mathbb{R}^n_{++}$ $: M\left(y_1, y_2\right) > M\left(y_1^{\delta(ij)}, y_2\right).$

SSM.3 For all $i, j \in N$ with $y_{1l} \geq y_{1k} > \delta_1 \in \mathbb{R}_{++}$, $y_{2j} \geq y_{2i} > \delta_2 \in \mathbb{R}_{++}$, all $y_1 \in R^n_{++}(kl)$ and all $y_2 \in R^n_{++}(ij) : M\left(y_1^{\delta_1(kl)}, y_2^{\delta_2(ij)}\right) > M\left(y_1, y_2\right).$

SSM.2 (SSM.1) says that covariance-neutral inequality increasing transfers in the second (first) period income distribution increase (decrease) the mobility index[4]. SSM.3 says that covariance-neutral inequality increasing transfers in both periods' income distribution increase the value of the mobility index. The implications of these axioms for a general mobility index of the form $M\left(y_1, y_2\right)$ are not clear. In the next section we impose more structure on the mobility index using the RISC axioms.

3 Results

Applying insights similar to Foster and Shorrocks (1991) we obtain the following representation result.

Lemma 1 *: **Replication Invariant Subgroup Consistent (RISC) Mobility Measures.***

For each $n \geq 1$ and every $(y_1, y_2) \in \mathbb{R}^{2n}_{++}$ a replication invariant subgroup consistent mobility measure can be written as

$$F\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(y_{1i}, y_{2i}\right)\right), \tag{1}$$

where $F : \phi\left(\mathbb{R}^2_{++}\right) \rightarrow \mathbb{R}$ is continuous and increasing and $\phi : \mathbb{R}^2_{++} \rightarrow \mathbb{R}$ is continuous.

[4]The interpretation here is analogous to the interpretation of the "within-type progressive transfer" in the cardinal variable considered by Gravel and Moyes (2008) in the context of bidimensional inequality measurement with an ordinal variable. In our framework both variables (income) are cardinal, and so we can consider transfers in both dimensions.

Particular subgroup consistent mobility measures have been proposed by Fields and Ok (1996, 1999) , namely

$$M_{FO_1} = \frac{1}{n}\sum_{i=1}^{n} |y_{2i} - y_{1i}|^{\alpha}, \quad M_{F0_2} = \frac{1}{n}\sum_{i=1}^{n} \left|\log\left(\frac{y_{2i}}{y_{1i}}\right)\right|^{\alpha} \text{ and } M_{FO_3} = \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{y_{2i}}{y_{1i}}\right),$$

where $\alpha \in \mathbb{R}_{++}$,

$$M_{FO_4} = \frac{1}{n}\sum_{i=1}^{n} \left|\frac{(y_{2i})^{1-\sigma}}{1-\sigma} - \frac{(y_{1i})^{1-\sigma}}{1-\sigma}\right| \text{ for } 0 \leq \sigma \neq 1 \text{ and } M_{FO_4} = \frac{1}{n}\sum_{i=1}^{n} \left|\log\left(\frac{y_{2i}}{y_{1i}}\right)\right| \text{ for } \sigma = 1,$$

where $\sigma \in \mathbb{R}_{++}$ and by D'Agostino and Dardanoni (2007),

$$M_{D_1} = \frac{1}{n}\sum_{i=1}^{n} (y_{1i} - y_{2i})^2 \text{ and } M_{D_2} = \frac{1}{n}\sum_{i=1}^{n} (g(y_{1i}) - g(y_{2i}))^2,$$

where $g(.)$ is a continuous and increasing function.[5]

Corollary 1 *$M_{FO_1}, M_{F0_2}, M_{FO_3}, M_{FO_4}, M_{D_1}$ and M_{D_2} are RISC measures of mobility. Their properties in terms of our core axioms (EM and FSM) and the secondary SSM axioms are collected in the following table:*

Table 1: **Some existing RISC Measures and their Properties**

measure satisfies axiom ?	M_{FO_1}	M_{F0_2}	M_{FO_3}	M_{FO_4}	M_{D_1}	M_{D_2}
EM	No	No	No	No	Yes	Yes
FSM	No	No	Yes	No	No	No
SSM.1	No	No	No	No	No	No
SSM.2	No	No	No	No	No	No
SSM.3	No	No	No	No	No	No

The measures proposed by D'Agostino and Dardanoni satisfy EM, but only M_{FO_3} satisfies FSM. Especially the fact that none of these measures satisfies simultaneously EM and FSM is worrying. None of the measures satisfies any of the SSM axioms. The following corollary shows the restrictions that are imposed by EM, FSM and SSM on replication invariant subgroup consistent mobility measures.

Corollary 2 *: **RISC Mobility Measures satisfying EM, FSM and SSM.***

A replication invariant subgroup consistent mobility can be written in the form (1) of Lemma 1. Moreover, it satisfies

[5] Mitra and Ok (1998) characterize the measure M_{FO_1} for values of $\alpha \geq 1$. The range or the value of α is not important for what follows, however. The same is true for the value of σ in M_{FO_4}, or the exact shape of the function $g(.)$ in M_{D_2}.

(a) *EM if and only if for all* $y_{1i}, y_{2i}, y_{1j}, y_{2j} \in \mathbb{R}^n_{++} : (y_{1i} - y_{1j})(y_{2i} - y_{2j}) > 0 \Rightarrow \phi(y_{1i}, y_{2i}) + \phi(y_{1j}, y_{2j}) < \phi(y_{1i}, y_{2j}) + \phi(y_{1j}, y_{2i})$,

(b) *FSM if and only if the function* ϕ *is decreasing in its first argument and increasing in its second argument,*

(c) *SSM.1 if and only if for all* $a, b, c, \delta \in \mathbb{R}_{++}$ *with* $\delta < b < c : \phi(a, b - \delta) - \phi(a, b) > \phi(a, c) - \phi(a, c + \delta)$,

(d) *SSM.2 if and only if for all* $a, b, c, \delta \in \mathbb{R}_{++}$ *with* $\delta < b < c : \phi(b - \delta, a) - \phi(b, a) < \phi(c, a) - \phi(c + \delta, a)$,

(e) *SSM.3 if and only if for all* $a, b, c, \delta \in \mathbb{R}_{++}$ *it satisfies the condition given under (c) and for all* $a, b, c, \delta \in \mathbb{R}_{++} : \phi(b - \delta, a) - \phi(b, a) > \phi(c, a) - \phi(c + \delta, a)$.

If the cross derivative of the function ϕ exists, the condition in part (a) of the Corollary is equivalent to requiring this cross derivative to be negative.[6] Condition (b) is self evident. If the function ϕ is twice differentiable the other conditions reduce to the following: (c) requires that the second derivative with respect to the second argument is positive, (d) that the second derivative with respect to the first argument is negative, and (e) that both second derivatives are positive.

A specific functional form for a mobility measure satisfying EM and FSM can be obtained by imposing two additional standard axioms for relative mobility measurement:

RSI [Ratio-scale Invariance] For all $y_1, y_2, x_1, x_2 \in \mathbb{R}^n_{++}$ and for all $\lambda_1, \lambda_2 \in \mathbb{R}_{++} : M(y_1, y_2) = M(x_1, x_2) \Leftrightarrow M(\lambda_1 y_1, \lambda_2 y_2) = M(\lambda_1 x_1, \lambda_2 x_2)$.

SI [Scale Invariance] For all $y_1, y_2 \in \mathbb{R}^n_{++}$ and for all $\lambda \in \mathbb{R}_{++} : M(\lambda y_1, \lambda y_2) = M(y_1, y_2)$.

We characterize a new mobility measure in the theorem 1 and state its properties with respect to the second structural mobility axioms in the subsequent corollary.

Theorem 1 *: A new index of relative mobility.*
A replication invariant, subgroup consistent mobility index satisfies EM, FSM, RSI and SI if and only if for each $n \geq 1$ *and every* $(y_1, y_2) \in \mathbb{R}^{2n}_{++}$ *it can be written as*

$$F\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_{2i}}{y_{1i}}\right)^{r}\right), \tag{2}$$

where $F : \phi\left(\mathbb{R}^2_{++}\right) \to \mathbb{R}$ *is continuous and increasing and* $r \in \mathbb{R}_{++}$.

Corollary 3 *The measure characterized in theorem 1 satisfies SSM.1 and SSM.2 if and only if* $r > 1$. *It never satisfies SSM.3.*

The class of mobility measures in Theorem 1 is new. A particular subclass, only indexed by r, follows by letting F be the identity map,

$$M(r) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_{2i}}{y_{1i}}\right)^{r}. \tag{3}$$

[6] In the context of evaluating exchange mobility by an additively separable dynastic social welfare function, this property was already established by Atkinson (1981) and Markandya (1982).

The parameter r has the interpretation of a sensitivity parameter. This becomes plain if M_r is rewritten as $\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_{2i}}{y_{1i}}\right)^{r-1}\left(\frac{y_{2i}}{y_{1i}}\right)$, where $\left(\frac{y_{2i}}{y_{1i}}\right)^{r-1}$ is the weight given to the relative income change of each individual. For $r=1$ all changes get the same weight. If $r<(>)\,1$, the weight for small changes is larger (smaller) than for big changes; SSM.1 or SSM.2 require that $r>1$.

Some of the measures proposed in the literature, such as M_{FO_1} and M_{D_1}, are not relative, but absolute measures of mobility. It is worth pointing out that the present framework can be easily adjusted to characterise an absolute measure of income mobility satisfying the RISC axioms EM and FSM by replacing $\mathbb{R}^n_{++}$ by $\mathbb{R}^n$ in all domain definitions, and after definition of ι as an n-dimensional vector of ones, by replacing RSI and SI by, respectively,

TSI [Translation-scale Invariance] For all $y_1, y_2, x_1, x_2 \in \mathbb{R}^n$ and for all $\kappa_1, \kappa_2 \in \mathbb{R}$: $M(y_1, y_2) = M(x_1, x_2) \Leftrightarrow M(y_1 + \kappa_1\imath, y_2 + \kappa_2\imath) = M(x_1 + \kappa_1\imath, x_2 + \kappa_2\imath)$.

AI [Addition Invariance] For all $y_1, y_2 \in \mathbb{R}^n$ and for all $\kappa \in \mathbb{R}_{++}$: $M(y_1 + \kappa\iota, y_2 + \kappa\iota) = M(y_1, y_2)$.

The result is stated in the following theorem.

Theorem 2 ***: A new index of absolute mobility.***

A replication invariant, subgroup consistent mobility index satisfies EM, FSM, TSI and AI if and only if for each $n \geq 1$ and every $(y_1, y_2) \in \mathbb{R}^{2n}$ it can be written as

$$F\left(\frac{1}{n}\sum_{i=1}^{n}\exp\left[c\left(y_{2i}-y_{1i}\right)\right]\right), \tag{4}$$

where $F : \phi\left(\mathbb{R}^2_{++}\right) \to \mathbb{R}$ is continuous and increasing and $c \in \mathbb{R}_{++}$.

Corollary 4 *The measure characterised in theorem 2 satisfies SSM.3 (and hence SSM.1). It never satisfies SSM.2.*

A particular subclass, only indexed by c, follows by letting F be the identity map,

$$m(c) = \frac{1}{n}\sum_{i=1}^{n}\exp\left[c\left(y_{2i}-y_{1i}\right)\right]. \tag{5}$$

Again c is a sensitivity parameter; larger values of c increase the effect of large absolute income movements. The measure of absolute income mobility always prefers an increase in dispersion in the marginal distributions of both periods, and so fails to satisfy SSM.2. Moreover to compare the evolution of mobility over time or between countries, relative mobility measures are more attractive, which explains why in the empirical section also our focus is on our relative mobility measure (3).

For empirical work, and specifically the empirical illustration that follows, we need to consider the issue of statistical inference for the mobility measure (3). The weak law of large numbers implies that as $n \to \infty$, $M(r) \to E\left((y_2/y_1)^r\right)$ and the central limit theorem implies that $M(r)$ is asymptotically distributed as a normal variate with mean μ_M and variance σ^2_M: $M(r) \sim^a N\left(\mu_M, \sigma^2_M\right)$. Comparing two independent joint distributions, such as in the context of cross-country comparisons, the standardised difference-of-means statistic is asymptotically distributed as a standard normal variate. Similar arguments apply to the measure of absolute mobility $m(c)$.

We therefore have the following:

Lemma 2 *: **Statistical inference for the mobility measures** $M(r)$ **and** $m(c)$.*

(a) $M(r) \sim^a N\left(\mu_M, \sigma_M^2\right)$ *with* $\mu_M = E\left((y_2/y_1)^r\right)$ *and* $n\sigma_M^2 = E\left((y_2/y_1)^{2r}\right) - \left[E\left((y_2/y_1)^r\right)\right]^2$. *Both* μ_M *and* σ_M^2 *can be consistently estimated by their sample analogues.*

(b) $m(c) \sim^a N\left(\mu_m, \sigma_m^2\right)$ *with* $\mu_m = E\left(\exp\left[c\left(y_2 - y_1\right)\right]\right)$ *and* $n\sigma_m^2 = E\left(\exp\left[2c\left(y_2 - y_1\right)\right]\right) - \left[E\left(\exp\left[c\left(y_2 - y_1\right)\right]\right)\right]^2$. *Both* μ_m *and* σ_m^2 *can be consistently estimated by their sample analogues.*

(c) Consider two independent joint income distributions and the associated mobility measures $M_1(r)$ *and* $M_2(r)$ *with* $M_1(r) \sim^a N\left(\mu_{M,1}, \sigma_{M,1}^2\right)$ *and* $M_2(r) \sim^a N\left(\mu_{M,2}, \sigma_{M,2}^2\right)$. *Under the null hypothesis that* $\mu_{M,1} = \mu_{M,2}$ *we have* $\left[M_1(r) - M_2(r)\right] / \left[\sigma_{M,1}^2 + \sigma_{M,2}^2\right]^{1/2} \sim^a N(0,1)$.

(d) Consider two independent joint income distributions and the associated mobility measures $m_1(c)$ *and* $m_2(c)$. *To test the hypothesis that* $\mu_{m,1} = \mu_{m,2}$ *apply (c) with* M *replaced by* m.

4 Empirical Illustration: Income Mobility in the USA and Germany Revisited

Our empirical application is placed in the context of the ongoing debate about income mobility comparisons between the USA and Germany (see e.g. Burkhauser et al, 1997a and 1997b, Maasoumi and Trede, 2001, Gottschalk and Spolaore, 2002, and Schluter and Trede, 2003).

Germany is a useful choice for comparison with the United States: the two countries are the largest and third largest economies but some key institutions differ. In contrast to the United States, the German labor market is characterized by rigidity and relatively centralized wage bargaining. The German social welfare system is much more generous. Hence, the received wisdom is that Germany exhibits both lower income inequality and lower income mobility. Burkhauser et al. (1997a, 1997b) have observed that when measuring income mobility using Shorrocks (1978) indices Germany is typically ranked *more mobile* than the US, contrary to this received wisdom. Gottschalk and Spolaore (2002) and Schluter and Trede (2003) have advanced some explanation for this surprising ranking. The application of our new measure makes a substantive complementary empirical contribution to this debate since we are able to explicitly highlight the contributions of structural and exchange mobility.[7]

We follow this empirical literature in the use of data sources, sample selection, and income definitions. The data are from the "Equivalent Data Files"[8] versions of the US Panel Study of Income Dynamics (PSID) and the German Socio-Economic Panel (GSOEP). Both panels are similar in design, and the data provider has generated comparable income variables. In order to be fully comparable to the literature cited above, we consider the same case as Schluter and Trede (2003): the unit of analysis is the person, the income concept

[7] Moreover, in the light of the discussion in Schluter and Trede (2003) our measure is also transparent about the "local" aspects of mobility, since our measure is subgroup consistent and thus explicit about the aggregation rule for income changes.

[8] http://www.human.cornell.edu/che/PAM/Research/Centers-Programs/German-Panel/cnef.cfm.

is net (i.e. post-tax post-benefit) income equivalised using the OECD scale (equal to the square root of the household size) in 1996 prices. The period under scrutiny are the years 1984 to 1992, when both countries went through a largely synchronised business cycle, and we consider annual income mobility, i.e. years t and $t+1$. For comparability and statistical robustness, we follow the literature and trim each sample at the 1% and 99% quantile. The resulting samples are in excess of 10,000 persons. We are able to replicate the results of cited literature, in particular Table 1 of Schluter and Trede (2003): the class of Shorrocks (1978) measures ranks Germany more mobile than the US. [9]

Before discussing our point mobility measures we present some descriptive statistics for the period 1987 and 1988; these turn out to be representative for all periods under investigation. In Figure 1 columns 1 and 2 we present contour plots of kernel density estimates of the joint and conditional income distributions (see Schluter, 1998, for similar estimates).[10] The US densities are more dispersed than the German counterparts. A particular (constant) feature of the conditional densities is the greater upward mobility of low-income Germans. Column 3 of the figure depicts the density estimate of relative incomes. This is of interest since the mobility index $M(r)$ transforms relatives incomes by means of the function $g(x) = x^r$, which is concave if $0 < r < 1$ and convex for $r \geq 1$. It is evident from the plots that the US density has far more mass in both tails than the density for Germany. Hence we expect that the $M(r)$ index ranks the US more mobile than Germany for $r \geq 1$, but this ranking could be reversed for r sufficiently small to compensate for the higher US mean.

We proceed to apply our mobility measure (3) for increasing values of the sensitivity parameter $r \in \{0.2, 0.4, 0.7, 1, 1.5, 2\}$. The results are reported in Table 2. In particular, Panels A and B report the point measures as well as the estimated standard errors. Panel C summarises the results, reporting whether the US is ranked more mobile than Germany, and indicates if the ranking is statistically not significant.

As expected from the plot of the relative income densities the measure $M(r)$ ranks the US more mobile than Germany for all values of $r \geq 1$ and all these difference are statistically significant. For smaller values of r the US is ranked more mobile than Germany for many but not all years; prominent exceptions are the periods 1986/87 and 1990/91 and, as expected, these exceptions become more prominent for smaller values of r. We conclude that the US typically exhibits more joint upward structural and exchange mobility than Germany. Mobility rankings based on the Shorrocks (1978) class are based on a measure which violates these notions.

5 Conclusions

We have derived axiomatically new mobility measures which respect the notions of structural and exchange mobility, and developed methods of statistical inference. We have illustrated the usefulness of the measure by considering the ongoing debate about mobility comparisons between the US and Germany. Our substantive empirical contribution is the insight that the US exhibits typically greater structural and exchange mobility, and hence the proposed measure $M(r)$ ranks the US typically more mobile than Germany. The empirical literature

[9] Precise details are not reproduced for reasons of brevity but are available on requests from the authors.

[10] Incomes have been normalised to for the sake of comparability, the (common) bandwidths have been chosen subjectively, and the conditional density is obtained by simply dividing the joint density estimate by an estimate of the marginal density.

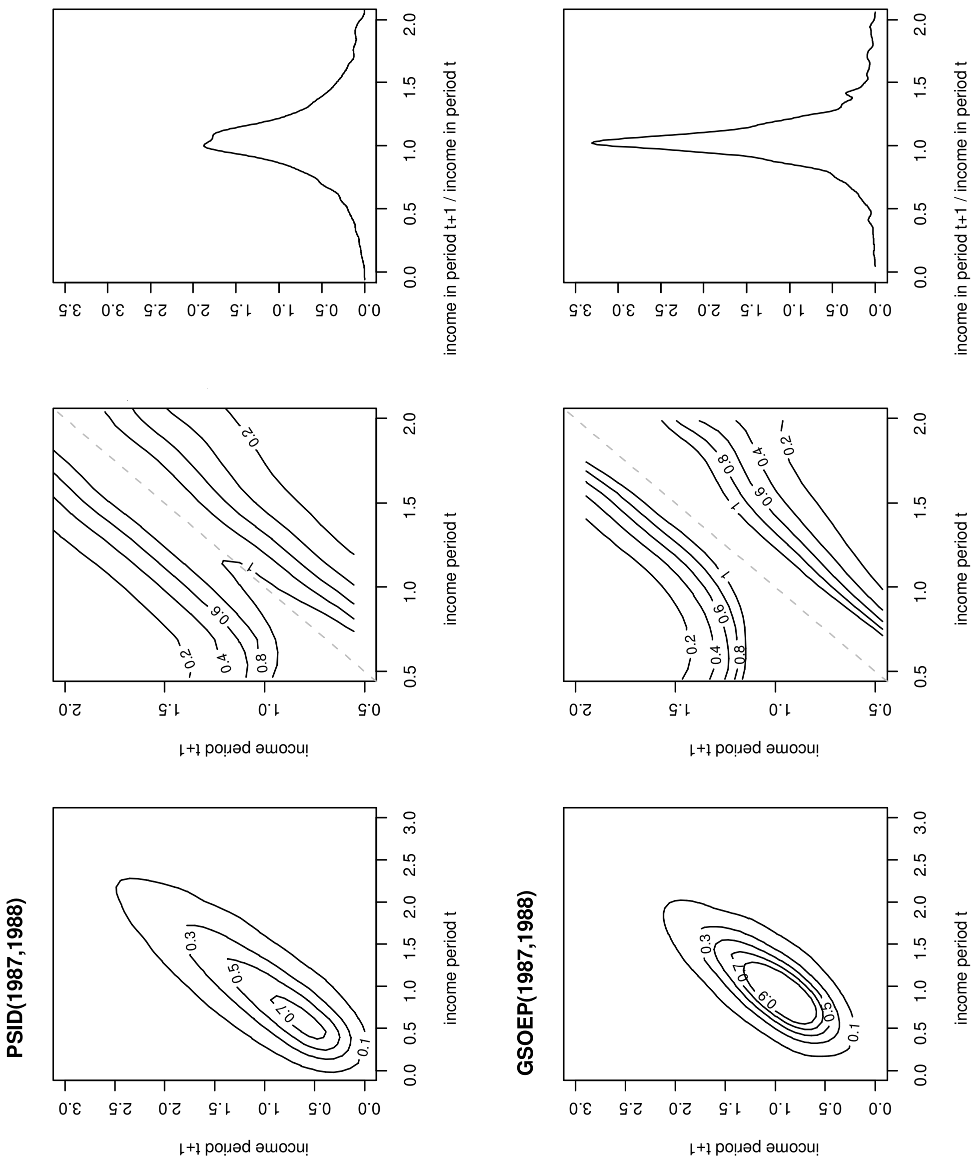
PSID(1987,1988)
GSOEP(1987,1988)
income period t
income period t+1
income in period t+1 / income in period t

Figure 1:

period	$M(.2)$	$M(.4)$	$M(.7)$	$M(1)$	$M(1.5)$	$M(2)$
			PSID			
1984/85	1.009	1.027	1.072	1.151	1.460	2.703
	(0.0007)	(0.0015)	(0.0035)	(0.0088)	(0.0570)	(0.4268)
1985/86	1.008	1.024	1.065	1.132	1.351	1.889
	(0.0007)	(0.0014)	(0.0030)	(0.0058)	(0.0194)	(0.0758)
1986/87	1.008	1.023	1.062	1.126	1.330	1.811
	(0.0006)	(0.0014)	(0.0029)	(0.0055)	(0.0173)	(0.0611)
1987/88	1.014	1.035	1.083	1.157	1.400	2.202
	(0.0006)	(0.0013)	(0.0030)	(0.0070)	(0.0440)	(0.3444)
1988/89	1.010	1.028	1.069	1.136	1.339	1.804
	(0.0006)	(0.0013)	(0.0028)	(0.0054)	(0.0167)	(0.0595)
1989/90	1.002	1.012	1.040	1.090	1.250	1.628
	(0.0006)	(0.0013)	(0.0026)	(0.0049)	(0.0175)	(0.0859)
1990/91	1.003	1.014	1.049	1.116	1.389	2.355
	(0.0007)	(0.0014)	(0.0034)	(0.0078)	(0.0386)	(0.2217)
1991/92	1.002	1.013	1.045	1.104	1.311	1.861
	(0.0007)	(0.0014)	(0.0030)	(0.0060)	(0.0213)	(0.0886)
			GSOEP			
1984/85	1.003	1.01	1.031	1.069	1.205	1.585
	0.0006	0.0013	0.0029	0.0059	0.0222	0.0945
1985/86	1.005	1.014	1.034	1.068	1.177	1.485
	(0.0006)	(0.0012)	(0.0026)	(0.0054)	(0.0237)	(0.1232)
1986/87	1.015	1.033	1.066	1.109	1.211	1.374
	(0.0005)	(0.0011)	(0.0021)	(0.0035)	(0.0081)	(0.0194)
1987/88	1.007	1.017	1.037	1.063	1.127	1.224
	(0.0005)	(0.0010)	(0.0018)	(0.0029)	(0.0058)	(0.0118)
1988/89	1.007	1.017	1.038	1.066	1.139	1.269
	(0.0005)	(0.0010)	(0.0020)	(0.0035)	(0.0090)	(0.0260)
1989/90	1.004	1.011	1.025	1.047	1.101	1.192
	(0.0005)	(0.0010)	(0.0019)	(0.0031)	(0.0065)	(0.0147)
1990/91	1.01	1.022	1.047	1.078	1.151	1.262
	(0.0005)	(0.0010)	(0.0020)	(0.0031)	(0.0066)	(0.0149)
1991/92	1.005	1.013	1.031	1.057	1.122	1.231
	(0.0005)	(0.0011)	(0.0021)	(0.0034)	(0.0071)	(0.0159)
			$M_{PSID}(r) > M_{GSOEP}(r)$			
1984/85	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
1985/86	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
1986/87	FALSE	FALSE	FALSE^{ns}	TRUE	TRUE	TRUE
1987/88	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
1988/89	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
1989/90	FALSE	TRUE^{ns}	TRUE	TRUE	TRUE	TRUE
1990/91	FALSE	FALSE	TRUE^{ns}	TRUE	TRUE	TRUE
1991/92	FALSE	FALSE^{ns}	TRUE	TRUE	TRUE	TRUE

Table 2: Mobility comparisons between the the US (PSID) and Germany (GSOEP). Notes: SE in parenthesis. In Panel 3 'ns' denotes not statistically significant at 5 percent level.

has employed measures which do not respect these notions.

We conclude by the methodological observation that some researchers (e.g. Van Kerm (2004), Ruiz-Castillo (2004)) seek to decompose overall mobility into a structural and exchange mobility component. When one wants more upward (downward) structural mobility to influence the index positively (negatively) such a decomposition should be based on a mobility index that satisfies both the EM and FSM axioms. Our measures are therefore prime candidate for such decomposition exercises.

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A Appendix: Proofs

Proof of lemma 1. The proof of this lemma follows Foster and Shorrocks (1991)' proof of their proposition 1. For completeness it is repeated here. We first establish that the mobility measure needs to be separable, defined as follows.

SEP [Separability]**:** For all $y_1^1, y_1^2, y_2^1, y_2^2, y_1^{1\prime}, y_1^{2\prime}, y_2^{1\prime}, y_2^{2\prime} \in \mathbb{R}_{++}^n$ and for all $P \in \mathcal{P}$:
$M\left(y_1^1, y_1^2, y_2^1, y_2^2\right) \geq M\left(y_1^{1\prime}, y_1^2, y_2^{1\prime}, y_2^2\right) \Rightarrow M\left(y_1^1, y_1^{2\prime}, y_2^1, y_2^{2\prime}\right) \geq M\left(y_1^{1\prime}, y_1^{2\prime}, y_2^{1\prime}, y_2^{2\prime}\right).$

Result 1. If a mobility measure is subgroup consistent and symmetric then it also satisfies SEP.

Proof. Suppose $M\left(y_1^1, y_1^2, y_2^1, y_2^2\right) \geq M\left(y_1^{1\prime}, y_1^2, y_2^{1\prime}, y_2^2\right)$. Then, by subgroup consistency we must have that $M\left(y_1^1, y_2^1\right) \geq M\left(y_1^{1\prime}, y_2^{1\prime}\right)$.

a) If $M\left(y_1^1, y_2^1\right) > M\left(y_1^{1\prime}, y_2^{1\prime}\right)$, then immediately by subgroup consistency $M\left(y_1^1, y_1^{2\prime}, y_2^1, y_2^{2\prime}\right) > M\left(y_1^{1\prime}, y_1^{2\prime}, y_2^{1\prime}, y_2^{2\prime}\right)$.

b) If $M\left(y_1^1, y_2^1\right) = M\left(y_1^{1\prime}, y_2^{1\prime}\right)$, we cannot have that $M\left(y_1^1, y_1^{2\prime}, y_2^1, y_2^{2\prime}\right) <$ $M\left(y_1^{1\prime}, y_1^{2\prime}, y_2^{1\prime}, y_2^{2\prime}\right)$, because this would imply by subgroup consistency that $M\left(y_1^1, y_1^{2\prime}, y_1^{1\prime}, y_2^1, y_2^{2\prime}, y_2^{1\prime}\right) < M\left(y_1^{1\prime}, y_1^{2\prime}, y_1^1, y_2^{1\prime}, y_2^{2\prime}, y_2^1\right)$, which contradicts the anonymity of $M\left(y_1, y_2\right)$. ■

Returning to the proof of lemma 1, let $z_i = [y_{1i}, y_{2i}]$. Then, using result 1, due to the fact that, by definition, $M\left(y_1, y_2\right)$ is not constant on D and a standard result on separability (due to Gorman (1968)), structural mobility, subgroup consistent, anonymous and continuous mobility measures can be written in the following form for any integer $n \geq 3$:

$$M\left(y_1, y_2\right) = \widetilde{F}_n\left(\sum_{i=1}^n \widetilde{\phi}_n\left(y_{1i}, y_{2i}\right)\right) \text{ for all } \left(y_1, y_2\right) \in \mathbb{R}_{++}^{2n},$$

where $\widetilde{F}_n$ is continuous and increasing and $\widetilde{\phi}_n$ is continuous.

Next, define $\phi_n\left(t\right) \equiv n\left[\widetilde{\phi}_n\left(t\right) - \widetilde{\phi}_n\left(v\right)\right]$ for $t \in \mathbb{R}_{++}^2 \Rightarrow \widetilde{\phi}_n\left(t\right) = \frac{1}{n}\phi_n\left(t\right) + \widetilde{\phi}_n\left(v\right)$, $\Rightarrow \sum_{i=1}^n \widetilde{\phi}_n\left(z_i\right) = \frac{1}{n}\sum_{i=1}^n \phi_n\left(z_i\right) + n\widetilde{\phi}_n\left(v\right)$. Define $F_n\left(u\right) \equiv \widetilde{F}_n\left(u + n\widetilde{\phi}_n\left(v\right)\right)$ for all $u \in \phi_n\left(\mathbb{R}_{++}^2\right)$, hence

$\widetilde{F}_n\left(\sum_{i=1}^n \widetilde{\phi}_n\left(z_i\right)\right) = \widetilde{F}_n\left(\frac{1}{n}\sum_{i=1}^n \phi_n\left(z_i\right) + n\widetilde{\phi}_n\left(v\right)\right) = F_n\left(\frac{1}{n}\sum_{i=1}^n \phi_n\left(z_i\right)\right)$,

such that we can write subgroup consistent, anonymous and continuous mobility measures as

$$M\left(y_1, y_2\right) = F_n\left(\frac{1}{n}\sum_{i=1}^n \phi_n\left(y_{1i}, y_{2i}\right)\right) \text{ for all } n \geq 3 \text{ and } \left(y_1, y_2\right) \in \mathbb{R}_{++}^{2n},$$

where $F_n : \phi_n\left(\mathbb{R}_{++}^2\right) \to \mathbb{R}$ is continuous and increasing and $\phi_n : \mathbb{R}_{++}^2 \to \mathbb{R}$ is continuous .

Note that replication invariance allows us to choose the functions ϕ_n and F_n to be independent of n, and extend the formula to the cases where $n = 1$ and $n = 2$. This is shown next.

Take a particular 2-dimensional vector t and replicate it 4 times to obtain the vector w. Denote the $m = 4n$ (with n a positive integer) times replication of vector t by w'. Define $\phi \equiv \phi_4$ and $F \equiv F_4$. By RISC.2, $F_m\left(\phi_m\left(t\right)\right) = M\left(w'\right) = M\left(w\right) = F\left(\phi\left(t\right)\right)$ for every $t \in \mathbb{R}_{++}^2$, such that $\phi_m\left(t\right) = F_m^{-1}\left(F\left(\phi\left(t\right)\right)\right)$.

Consequently, mobility for (y_1, y_2) with y_1 and y_2 of dimension m becomes $M(y_1, y_2) = F_m\left(\frac{1}{m}\sum_{i=1}^{m}\phi_m(z_i)\right) = F_m\left(\frac{1}{m}\sum_{i=1}^{m}F_m^{-1}(F(\phi(z_i)))\right)$, and so $F^{-1}(M(y_1, y_2)) =$
$F^{-1}\left(F_m\left(\frac{1}{m}\sum_{i=1}^{m}F_m^{-1}(F(\phi(z_i)))\right)\right)$, which can be written as

(A1) $$F^{-1}(M(y_1, y_2)) = G_m^{-1}\left(\frac{1}{m}\sum_{i=1}^{m}G_m(\phi(y_{1i}, y_{2i}))\right),$$

after defining $G_m^{-1}(u) = F^{-1}(F_m(u))$, a continuous and increasing function on $\phi\left(\mathbb{R}^2_{++}\right)$ and since $F = F_4$, $G_4(u) = u$.

Now consider any $(y_1, y_2) \in \mathbb{R}^2_{++}$ and its replications $(y_1', y_2') \in \mathbb{R}^{4\cdot 2}_{++}$ and $(y_1'', y_2'') \in \mathbb{R}^{m\cdot 2}_{++}$. Define $u_1 = \phi(z_1) = \phi(y_{11}, y_{21})$ and $u_2 = \phi(z_2) = \phi(y_{12}, y_{22})$. We then have from (A1)
$F^{-1}(M(y_1'', y_2'')) = G_m^{-1}\left(\frac{1}{m}\sum_{i=1}^{m}G_m(\phi(z_i''))\right) = G_m^{-1}\left(\frac{1}{2}G_m(u_1) + \frac{1}{2}G_m(u_2)\right)$ and from, first RISC.2 and then (A1) again,
$F^{-1}(M(y_1'', y_2'')) = F^{-1}(M(y_1', y_2')) = G_4^{-1}\left(\frac{1}{2}G_4(u_1) + \frac{1}{2}G_4(u_2)\right) = \frac{1}{2}(u_1 + u_2)$,
such that since both LHS are equal,

$$\left(\frac{1}{2}G_m(u_1) + \frac{1}{2}G_m(u_2)\right) = G_m\left(\frac{1}{2}(u_1 + u_2)\right) \text{ for all } u_1, u_2 \in \phi\left(\mathbb{R}^2_{++}\right)$$

The solution to this Jensen equation (Aczél (1966), p.46) implies $G_m(u) = r_m u + s_m$ for some constants r_m and s_m, which upon substitution in
$F^{-1}(M(y_1, y_2)) = G_m^{-1}\left(\frac{1}{m}\sum_{i=1}^{m}G_m(\phi(z_i))\right)$
$= \frac{1}{r_m}\left[\frac{1}{m}\sum_{i=1}^{m}(r_m\phi(z_i) + s_m) - s_m\right]$
$= \frac{1}{r_m}\left[r_m\frac{1}{m}\sum_{i=1}^{m}\phi(z_i) + s_m - s_m\right]$
$= \frac{1}{m}\sum_{i=1}^{m}\phi(z_i)$,
so we have

$$F^{-1}(M(y_1, y_2)) = \frac{1}{m}\sum_{i=1}^{m}\phi(y_{1i}, y_{2i}) \text{ for all } (y_1, y_2) \in \mathbb{R}^{m\cdot 2}_{++}$$

whenever $m = 4n$ and n is a positive integer.

Finally, for each $n \geq 1$ consider any $(y_1, y_2) \in \mathbb{R}^{2n}_{++}$ and its replication $(y_1', y_2') \in \mathbb{R}^{4\cdot 2n}_{++}$. By RISC.2 and the last equation, we obtain $F^{-1}(M(y_1, y_2)) = F^{-1}(M(y_1', y_2')) = \frac{1}{4n}\sum_{i=1}^{4n}\phi(z_i') = \frac{1}{n}\sum_{i=1}^{n}\phi(z_i)$, such that

$$M(y_1, y_2) = F\left(\frac{1}{n}\sum_{i=1}^{n}\phi(y_{1i}, y_{2i})\right) \text{ for each } n \geq 1 \text{ and every } (y_1, y_2) \in \mathbb{R}^{2n}_{++},$$

where $F : \phi\left(\mathbb{R}^2_{++}\right) \to \mathbb{R}$ is continuous and increasing and $\phi : \mathbb{R}^2_{++} \to \mathbb{R}$ is continuous. ■

Proof of corollary 2. The proof follows directly from lemma 1 and axioms EM, FSM and SSM. ■

Proof of theorem 1. The following proof is a completed version[11] of the proof of theorem 1 in Tsui (1995). We first need two lemmas.

[11] We are grateful to Thomas Demuynck for the proof of lemma 3.

Lemma 2. $M(y_1, y_2) \geq M(y_1', y_2')$ if and only if, for all $\lambda_1, \lambda_2 \in \mathbb{R}_{++} : M(\lambda_1 y_1, \lambda_2 y_2) \geq M(\lambda_1 y_1', \lambda_2 y_2')$.

Proof. Assume on the contrary that

$M(y_1, y_2) \geq (\leq) M(y_1', y_2')$ and $M(\lambda_1 y_1, \lambda_2 y_2) < (>) M(\lambda_1 y_1', \lambda_2 y_2')$.

The function $M(\delta_1 y_1, \delta_2 y_2)$ is a continuous function of δ_1 and $\delta_2 \in \mathbb{R}_{++}$. By assumption the image of the function contains positive and negative values. Since the function is continuous, by the intermediate value theorem, there must exist numbers γ_1 and $\gamma_2 \in \mathbb{R}_{++}$ such that $M(\gamma_1 y_1, \gamma_2 y_2) = M(\gamma_1 y_1', \gamma_2 y_2')$.

Using scale invariance twice,

$M(\lambda_1 y_1, \lambda_2 y_2) = M(\gamma_1 y_1, \gamma_2 y_2)$ and $M(\lambda_1 y_1', \lambda_2 y_2') = M(\gamma_1 y_1', \gamma_2 y_2')$,

such that $M(\lambda_1 y_1, \lambda_2 y_2) = M(\lambda_1 y_1', \lambda_2 y_2')$, a contradiction. ■

Lemma 3. For all $a_1, a_2, b_1, b_2, d_1, d_2, e_1, e_2 \in \mathbb{R}_{++}$ and all $\lambda_1, \lambda_2 \in \mathbb{R}_{++}$:

$$\text{(A2)} \qquad \frac{\phi(\lambda_1 a_1, \lambda_2 a_2) - \phi(\lambda_1 b_1, \lambda_2 b_2)}{\phi(a_1, a_2) - \phi(b_1, b_2)} = \frac{\phi(\lambda_1 d_1, \lambda_2 d_2) - \phi(\lambda_1 e_1, \lambda_2 e_2)}{\phi(d_1, d_2) - \phi(e_1, e_2)}.$$

Proof. Consider eight elements of $\mathbb{R}_{++}$: $a_1, a_2, b_1, b_2, d_1, d_2, e_1$ and e_2 and define the real number k such that:

$$\text{(A3)} \qquad \phi(d_1, d_2) - \phi(e_1, e_2) = k(\phi(a_1, a_2) - \phi(b_1, b_2)).$$

Case 1: k is a rational number. Consider two natural numbers K and L, such that $|k| = K/L$.

(a) If $k = K/L$, define $n = K + L$ and rewrite equation (A3) as:

$$\frac{K}{n}\phi(a_1, a_2) + \frac{L}{n}\phi(e_1, e_2) = \frac{K}{n}\phi(b_1, b_2) + \frac{L}{n}\phi(d_1, d_2).$$

Using ratio-scale invariance, this implies

$$\frac{K}{n}\phi(\lambda_1 a_1, \lambda_2 a_2) + \frac{L}{n}\phi(\lambda_1 e_1, \lambda_2 e_2) = \frac{K}{n}\phi(\lambda_1 b_1, \lambda_2 b_2) + \frac{L}{n}\phi(\lambda_1 d_1, \lambda_2 d_2).$$

Using the last two equations,

$$\frac{\phi(\lambda_1 a_1, \lambda_2 a_2) - \phi(\lambda_1 b_1, \lambda_2 b_2)}{\phi(\lambda_1 d_1, \lambda_2 d_2) - \phi(\lambda_1 e_1, \lambda_2 e_2)} = \frac{1}{k} = \frac{\phi(a_1, a_2) - \phi(b_1, b_2)}{\phi(d_1, d_2) - \phi(e_1, e_2)},$$

which after exchanging the numerator and denominator yields (A2).

(b) If $k = -K/L$, define $n = K + L$ and derive from (A3) that

$$\frac{K}{n}\phi(a_1, a_2) + \frac{L}{n}\phi(d_1, d_2) = \frac{K}{n}\phi(b_1, b_2) + \frac{L}{n}\phi(e_1, e_2).$$

Proceeding in the same way as under (a) allows us to show (A2).

Case 2: k is an irrational number.

(a) Suppose that $\phi(a_1, a_2) - \phi(b_1, b_2) > 0$.

(i) Suppose that $k > 0$. Consider two strictly positive arbitrary rational numbers r_1 and r_2 such that $r_1 < k < r_2$. Consequently, using (A3),

$$r_1(\phi(a_1, a_2) - \phi(b_1, b_2)) < \phi(d_1, d_2) - \phi(e_1, e_2) < r_2(\phi(a_1, a_2) - \phi(b_1, b_2)).$$

Let $r_1 = K_1/L_1$ and $r_2 = K_2/L_2$. Defining $n_1 = K_1 + L_1$ and $n_2 = K_2 + L_2$, we derive that

$$\begin{aligned}
\frac{K_1}{n_1}\phi(a_1,a_2) + \frac{L_1}{n_1}\phi(e_1,e_2) &< \frac{K_1}{n_1}\phi(b_1,b_2) + \frac{L_1}{n_1}\phi(d_1,d_2),\\
\frac{K_2}{n_2}\phi(a_1,a_2) + \frac{L_2}{n_2}\phi(e_1,e_2) &> \frac{K_2}{n_2}\phi(b_1,b_2) + \frac{L_2}{n_2}\phi(d_1,d_2).
\end{aligned}$$

Due to lemma 2, these inequalities imply

$$\begin{aligned}
\frac{K_1}{n_1}\phi(\lambda_1 a_1,\lambda_2 a_2) + \frac{L_1}{n_1}\phi(\lambda_1 e_1,\lambda_2 e_2) &< \frac{K_1}{n_1}\phi(\lambda_1 b_1,\lambda_2 b_2) + \frac{L_1}{n_1}\phi(\lambda_1 d_1,\lambda_2 d_2),\\
\frac{K_2}{n_2}\phi(\lambda_1 a_1,\lambda_2 a_2) + \frac{L_2}{n_2}\phi(\lambda_1 e_1,\lambda_2 e_2) &> \frac{K_2}{n_2}\phi(\lambda_1 b_1,\lambda_2 b_2) + \frac{L_2}{n_2}\phi(\lambda_1 d_1,\lambda_2 d_2).
\end{aligned}$$

Rearranging these equations we have that

$$r_1 = \frac{K_1}{L_1} < \frac{\phi(\lambda_1 d_1,\lambda_2 d_2) - \phi(\lambda_1 e_1,\lambda_2 e_2)}{\phi(\lambda_1 a_1,\lambda_2 a_2) - \phi(\lambda_1 b_1,\lambda_2 b_2)} < \frac{K_2}{L_2} = r_2. \tag{A4}$$

We proceed now to show that

$$\frac{\phi(\lambda_1 d_1,\lambda_2 d_2) - \phi(\lambda_1 e_1,\lambda_2 e_2)}{\phi(\lambda_1 a_1,\lambda_2 a_2) - \phi(\lambda_1 b_1,\lambda_2 b_2)} = k. \tag{A5}$$

Suppose this equality does not hold. In that case, there exists a rational number $r_1' > 0$ or a rational number $r_2' > 0$ such that either

(A6)

$$\frac{\phi(\lambda_1 d_1,\lambda_2 d_2) - \phi(\lambda_1 e_1,\lambda_2 e_2)}{\phi(\lambda_1 a_1,\lambda_2 a_2) - \phi(\lambda_1 b_1,\lambda_2 b_2)} < r_1' < k, \text{ or } \frac{\phi(\lambda_1 d_1,\lambda_2 d_2) - \phi(\lambda_1 e_1,\lambda_2 e_2)}{\phi(\lambda_1 a_1,\lambda_2 a_2) - \phi(\lambda_1 b_1,\lambda_2 b_2)} > r_2' > k.$$

However, since the rational numbers leading to (A4) were arbitrarily chosen, (A4) applies for r_1' and r_2', such that we must have

$$r_1' < \frac{\phi(\lambda_1 d_1,\lambda_2 d_2) - \phi(\lambda_1 e_1,\lambda_2 e_2)}{\phi(\lambda_1 a_1,\lambda_2 a_2) - \phi(\lambda_1 b_1,\lambda_2 b_2)} < r_2',$$

which contradicts (A6), such that (A5) must hold true. Combining (A5) with (A3) proofs (A2).

(ii) Suppose that $k < 0$. Consider two strictly negative arbitrary rational numbers r_1 and r_2 such that $r_1 < k < r_2$. Consequently, using (A3),

$$r_1(\phi(a_1,a_2) - \phi(b_1,b_2)) < \phi(d_1,d_2) - \phi(e_1,e_2) < r_2(\phi(a_1,a_2) - \phi(b_1,b_2)).$$

Let $r_1 = -K_1/L_1$ and $r_2 = -K_2/L_2$. Defining $n_1 = K_1 + L_1$ and $n_2 = K_2 + L_2$, we derive that

$$\begin{aligned}
\frac{K_1}{n_1}\phi(a_1,a_2) + \frac{L_1}{n_1}\phi(d_1,d_2) &> \frac{K_1}{n_1}\phi(b_1,b_2) + \frac{L_1}{n_1}\phi(e_1,e_2),\\
\frac{K_2}{n_2}\phi(a_1,a_2) + \frac{L_2}{n_2}\phi(d_1,d_2) &< \frac{K_2}{n_2}\phi(b_1,b_2) + \frac{L_2}{n_2}\phi(e_1,e_2).
\end{aligned}$$

Repeating the same steps as in (i), it is easy to show that (A2) holds true.

(b) Suppose that $\phi(a_1, a_2) - \phi(b_1, b_2) < 0$.

(i) Suppose that $k > 0$. Consider two strictly positive arbitrary rational numbers r_1 and r_2 such that $r_1 < k < r_2$. With $r_1 = K_1/L_1$, $r_2 = K_2/L_2$, $n_1 = K_1 + L_1$ and $n_2 = K_2 + L_2$, compared to the proof in part (a), (i), all inequalities before inequality (A4) switch sign, but (A4) remains true. One can further proceed as in (a), (i) to show that (A2) holds true.

(ii) Suppose that $k < 0$. Consider two strictly negative arbitrary rational numbers r_1 and r_2 such that $r_1 < k < r_2$. With $r_1 = -K_1/L_1, r_2 = -K_2/L_2, n_1 = K_1 + L_1$ and $n_2 = K_2 + L_2$, one can proceed as in part (a), (ii), paying heed to the signs of the first sets of inequalities, to show that (A2) holds true. ■

Returning to the proof of the theorem, put the right hand side equal to the number $R(\lambda_1, \lambda_2)$. Fixing (b_1, b_2), the equation can be written as

$\phi(\lambda_1 a_1, \lambda_2 a_2) - \phi(\lambda_1 b_1, \lambda_2 b_2) = R(\lambda_1, \lambda_2)[\phi(a_1, a_2) - \phi(b_1, b_2)] \Leftrightarrow$

$\phi(\lambda_1 a_1, \lambda_2 a_2) = R(\lambda_1, \lambda_2)\phi(a_1, a_2) + \phi(\lambda_1 b_1, \lambda_2 b_2) - R(\lambda_1, \lambda_2)\phi(b_1, b_2)$.

With (b_1, b_2) fixed, put $Q(\lambda_1, \lambda_2) = \phi(\lambda_1 b_1, \lambda_2 b_2) - R(\lambda_1, \lambda_2)\phi(b_1, b_2)$ and we get $\phi(\lambda_1 a_1, \lambda_2 a_2) = R(\lambda_1, \lambda_2)\phi(a_1, a_2) + Q(\lambda_1, \lambda_2)$. The solutions to this functional equation are, see Aczél et al. (1986):

$\phi(a_1, a_2) = a + b a_1^{-r_1} a_2^{r_2}$ and $\phi(a_1, a_2) = a - r_1 \ln(a_1) + r_2 \ln(a_2)$,

where $b > 0$, r_1 and $r_2 > 0$, to incorporate FSM. EM eliminates the logarithmic measure. The values of a and b do not matter since $F(.)$ is increasing. SI implies that $r_1 = r_2$, such that we can write the mobility index as in the theorem.

We still need to prove the independence of the axioms. As usual, we do this by proposing mobility measures that satisfy all axioms used, except one. The second column of the following table formulates measures that satisfy all axioms of the theorem, except the axiom indicated in the first column.

AXIOM	MEASURE
RISC.1 [Anonymity]	$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_{2i}}{y_{1i}}\right)^{r_i}$ with all $r_i > 0$, $r_i = 1$ if $\left(\frac{y_{2i}}{y_{1i}}\right) = \left(\frac{y_{21}}{y_{11}}\right)$ and $r_i \neq 1$ otherwise.
RISC.2 [Rep Invar]	$\sum_{i=1}^{n}\left(\frac{y_{2i}}{y_{1i}}\right)^{r}$ with $r > 0$.
RISC.3 [Subgr Con]	$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_{2i}}{\frac{1}{n}\sum_{j=1}^{n} y_{1j}}\right)^{r}$ with $r > 0$.
EM	$\frac{1}{n}\sum_{i=1}^{n}(\ln(y_{2i}) - \ln(y_{1i}))$.
FSM	$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_{1i}}{y_{2i}}\right)^{r}$ with $r > 0$.
RSI	$\frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{y_{2i}}{y_{1i}}\right)$.
SI	$\frac{1}{n}\sum_{i=1}^{n}(y_{2_i})^{r_2}(y_{1i})^{-r_1}$ with $r_1, r_2 > 0$.

■

Proof of theorem 2. Perform a transformation of all variables by putting $\widehat{y}_{1i} = \exp[y_{1i}]$ and $\widehat{y}_{2i} = \exp[y_{2i}]$. We are now in the positive domain, and TSI and AI reduce to RSI and SI. Hence we can apply Theorem 1 in terms of the transformed variables. Returning to the original variables, and putting $c = r$, completes the proof of the theorem.