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Note on State Dependent Mutations as an Equilibrium Refinement Device

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Abstract

This note focuses on the use of utility monotonic perturbations as a means of modelling the mutation process in evolutionary models. We show that a game has a detailed balanced and utility monotonic perturbation if and only if it is an ordinal potential game. We also show that utility monotonicity is not strong enough to serve as an equilibrium refinement device for ordinal potential games. An equilibrium refinement device that is applicable to a general class of games must, therefore, satisfy a stronger utility monotonicity condition while the detailed balance condition can no longer hold. We believe that a tightening of the bounds on the magnitude of stationary distributions could substantially further research in this topic.

1 Introduction

To overcome path dependence of the dynamic process, Kandori et al. [1993] and Young [1993] introduced noise (mutations, perturbations) in evolutionary models: a state is called stochastic stable if, as the mutation rate converges to zero, it is a limiting state of the process with mutation/perturbation. Both Kandori et al. [1993] and Young [1993] use a uniform mutation rate.

Their predictions, however, are criticized by Bergin and Lipman [1996] who show that by allowing the mutation rate to vary across states, *any* stable state of the model without mutation can be obtained as a stochastic stable state of a process with mutation. In other words, the equilibrium correspondence which maps the process with mutations to the set of stable states is surjective. Therefore, Blume [2003] draws on state dependent mutation

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rates to get a more profound understanding of the mutation process: he assumes that when an individual i prefers x to y, the probability that i will move from y to x is larger than the probability that i will move from x to y. We say that the mutation process exhibits *utility monotonicity*.

A more methodological critique on Kandori et al. [1993] and Young [1993] concerns the graph-theoretic principles which underly the results. The graph-methodology not only fails to add intuition and transparency to their analysis, more importantly, it does not allow for an exact computation of the limiting distribution of the perturbed process. Therefore, Markov chains which satisfy the detailed balance conditions are very interesting: they don't require the Markov chain tree theorem to find the stochastic stable states and it is straightforward to find their limiting distribution.

We argue that utility monotonicity alone is not strong enough to serve as an equilibrium refinement device. A stronger condition is required. Moreover, we show that such a stronger condition inevitably implies that the equilibrium refinement can not satisfy the detailed balance condition. We establish our argument in two steps: first, we demonstrate that a game has a detailed balanced and utility monotonic perturbation if and only if it is an ordinal potential game. Then, we show that for all ordinal potential games and all stable states of a deterministic evolutionary process, we can find a utility monotonic and detailed balanced perturbation such that the stochastic stable states coincide with those stable states. Utility monotonicity is, therefore, no solution to the Bergin and Lipman–critique.[1]

Remark, furthermore, that although it is a known result in the literature (cf. Young [1998]; Blume [2003]; Baron et al. [2003]) that the stochastic stable states of the detailed balanced and utility monotonic Markov chains of potential games are the states which maximize the potential function of the game, we know of no result that formally links potential games, utility monotonicity and detailed balancedness.

The paper is organized as follows: section 2 presents the necessary notation and definitions. In section 3 we demonstrate the link between ordinal potential games, utility monotonicity and detailed balancedness of the mutation process and we elaborate on the potential of state dependent mutation as an equilibrium refinement device in potential games. Finally, in section 4 we conclude by amplifying the implications of our results on the search for a general equilibrium refinement device.

2 Notation and Definitions

2.1 Markov Chains and Detailed Balancedness

Consider a finite set of states, S. A Markov chain, M, is a set of positive probabilities, $\{p(x,y)\}_{x,y\in S}$, such that for all $x \in S : \sum_{y\in S} p(x,y) = 1$.

[1]Note that, with a refinement of the allowed mutation processes, our result generalizes Bergin and Lipman [1996], while at the same time, by only considering ordinal potential games, our result is weaker than theirs. As such our result is neither stronger or weaker than theirs.

The element $p(x,y)$ is the transition probability of moving from state x to state y. Given a Markov chain, M, we define the binary relation $R_M \subseteq S \times S$, by $(x,y) \in R_M$ if and only if $p(y,x) > 0$. We denote by $T(R_M)$ the transitive closure of R_M.[2]

A set $A \subseteq S$ is a stable set of the Markov process M if (i) for all $x, y \in A$, $(x,y) \in T(R)$, and (ii) for all $x \in A, y \in S - A$, $(y,x) \notin R_M$; i.e. if for all $x, y \in A$ there is positive probability of going from x to y in finite time while the probability of going from a state in A to a state outside A is zero.

Each Markov chain has at least one stable set and two different stable sets must have nonempty intersection. The elements in the stable sets of M are called the stable states of the Markov process.

A Markov chain is said to be irreducible if and only if for all $x, y \in S$: $(x,y) \in T(R_M)$. An irreducible Markov chain has the property that the probability of going from any state in S to any other state in S in finite time is positive. The stable set of an irreducible Markov chain is unique and equals the entire set of states, S.

A probability distribution $\{P(x)\}_{x \in S}$ over S is an invariant distribution of the Markov chain M if for all $x \in S$:

$$\sum_{x \in S} P(x)p(x,y) = P(y).$$

We know that:

Fact 1. *Every Markov chain has an invariant distribution.*

Note that we will mainly be interested in the elements of S for which an invariant distribution $\{P(x)\}_{x \in S}$ has positive support, i.e. the set $\{x \in S \,|P(x) > 0\}$. The following results connect the stable sets and the support of an invariant distribution:

Fact 2. *If an invariant distribution $\{P(x)\}_{x \in S}$ of the Markov chain M has positive support on $x \in S$, then x is contained in a stable set of M.*

Fact 1 and 2 tell us that there exists at least one stable set. Furthermore, we know that:

Fact 3. *If M is an irreducible Markov chain, it has a unique invariant distribution, i.e. a unique stable set.*

Consider an invariant distribution $\{P(x)\}_{x \in S}$ of a Markov chain. We say that it satisfies the *detailed balance condition* if and only if for all $x, y \in S$:

$$P(y)p(y,x) = P(x)p(x,y).$$

A distribution which satisfies the detailed balance condition is automatically an invariant distribution. A Markov chain which has an invariant distribution that satisfies the detailed balance condition is said to be detailed balanced.

[2] $(x,y) \in T(R_M)$ if and only if there exists a sequence $x_1, \ldots, x_n$ of elements in S such that $x_1 = x, x_n = y$ and $\forall t = 1, \ldots, n-1 : (x_t, x_{t+1}) \in R_M$.

2.2 Perturbations induced by a Game

A game in strategic form, $G = (N, S, \{u_i\}_{i \in N})$, consists of a finite set of individuals $N = \{1, \dots, n\}$; a set of strategy profiles (states) $S = \prod_{i \in N} S_i$, where S_i is the finite set of strategies of individual i; and a set of functions $u_i : S \to \mathbb{R}$ which represent the payoffs to the individuals.

A strategy profile $s = (s_1, \dots, s_n)$ can be written as (s_i, s_{-i}), where s_{-i} represents the strategies of the individuals in $N - \{i\}$. For each individual i we define a binary relation $\approx_i \subseteq S \times S$ such that: $x \approx_i y$ if and only if $x_{-i} = y_{-i}$. Since $\approx_i$ is an equivalence relation (transitive, reflexive and symmetric), it partitions S into equivalence classes: for an element $x \in S$ we denote the equivalence class of x with respect to $\approx_i$ as $[x]_i$.

Consider the following discrete-time model. At each-time step one –random– individual $i \in N$ is given the opportunity to consider revising his strategy. He therefore maximizes his payoff, assuming that all the other individuals keep their strategy fixed. Formally, if at time t strategy s is played and at time $t+1$ individual i is selected, then at time $t+1$ strategy s' will be played, where $s' \in [s]_i$ and $s'_i \in \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$. We assume that when $\arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$ contains more than one element, each is chosen with equal probability.

This dynamic model defines a Markov chain $M_G = \{p(x, y)\}_{x,y \in S}$ with the set of state S. If there is an $i \in N$ such that $y \in \arg\max_{z \in [x]_i} u_i(z)$ then $p(x, y) > 0$, else $p(x, y) = 0$.

Consider an element $\delta \in \mathbb{R}_0^+$. A *perturbation* of the game G is a set of Markov chains $\{\Sigma_G(\delta)\}_{\delta \in \mathbb{R}_0^+}$ such that:

1. $\Sigma_G(.)$ is continuous in δ;
2. $\lim_{\delta \to \infty} \Sigma_G(\delta) = M_G$; and
3. $\Sigma_G(\delta)$ is irreducible for all $\delta \in \mathbb{R}_0^+$.

Let $\Sigma_G(\delta)$ be a perturbation of G. As $\Sigma_G(\delta)$ is irreducible, from fact 3 we know that it has a unique invariant distribution, P_δ, which has support on the whole of S. Moreover, as $\Sigma_G(.)$ is continuous in δ, P_δ is continuous in δ. A state $x \in S$ is said to be stochastic stable if $\lim_{\delta \to \infty} P_\delta(x) > 0$. The distribution $P_\infty = \lim_{\delta \to \infty} P_\delta$ is the stochastic stable distribution of the perturbation Σ_G. As P_δ is continuous, P_∞ is an invariant distribution of M_G. Therefore, every stochastic stable state is also a stable state: the set of stochastic stable states is a refinement of the set of stable states.

Remark that if a perturbation $\Sigma_G(\delta)$ of M_G satisfies the detailed balance condition, P_∞ necessarily satisfies the detailed balance condition for M_G.

2.3 Ordinal potential games and Utility Monotonicity

A game G is an *exact* potential game (henceforth, potential game) if there exists a function $V : S \to \mathbb{R}$ such that for all $i \in N, x \in S$ and $y \in [x]_i$:

$$u_i(x) - u_i(y) = V(x) - V(y).$$

Note that a potential function V of a potential game G is unique, up to a constant term (Monderer and Shapley [1996]).

A game G is an *ordinal* potential game[3] if there exists a function $V : S \to \mathbb{R}$ such that for all $i \in N$, $x \in S$ and $y \in [x]_i$:

$$u_i(x) - u_i(y) \geq 0 \leftrightarrow V(x) - V(y) \geq 0.$$

Consider a perturbation $\Sigma_G(\delta) = \{p_\delta(x,y)\}_{x,y \in S}$ of the game G. Following the idea in Blume [2003], we say that:

Definition 1. *$\Sigma_G(\delta)$ is utility monotonic if there exists a function $h : S \times S \to \mathbb{R}$ such that for all $i \in N$, $x \in S$ and $y \in [x]_i$:*

1. $\frac{p_\delta(y,x)}{p_\delta(x,y)} = e^{\delta h(x,y)}$, *and*
2. $h(x,y) \geq 0 \leftrightarrow u_i(x) - u_i(y) \geq 0$.

We call h a utility monotonic function for the perturbation Σ_G.

3 Results

The preceding definitions and notational elucidation now allow us to elaborate the core results of our paper, stated in theorem 1 and theorem 2.

Theorem 1. *The game G has a utility monotonic and detailed balanced perturbation if and only if G is an ordinal potential game.*

Proof. Let $\Sigma_G(\delta)$ be a detailed balanced and monotonic perturbation of G. Let $x \in S$ and $y, z \in [x]_i$. Let $\{P_\delta(x)\}_{x \in S}$ be the limiting distribution of $\Sigma_G(\delta) = \{p_\delta(x,y)\}_{x,y \in S}$ and consider the identity:

$$\frac{P_\delta(x)}{P_\delta(y)} \frac{P_\delta(y)}{P_\delta(z)} = \frac{P_\delta(x)}{P_\delta(z)}.$$

As $\Sigma_G(\delta)$ is utility monotonic and detailed balanced, we know that for all $i \in N$, $x \in S$ and $y, z \in [x]_i$: $h(x,z) = h(x,y) + h(y,z)$. With $x = y = z$, we derive that $h(x,x) = 0$; with $x = z$, we get: $h(x,y) = -h(y,x)$. Fix an element $z \in S$ and let $z_x^i = (z_i, x_{-i})$. Define the function $g(i,.)$ as: $g(i,x) = h(x, z_x^i)$. Then:

$$\begin{aligned} h(x,y) &= h(x, z_x^i)) + h(z_x^i, y). \\ &= h(x, z_x^i) - h(y, z_x^i). \\ &= g(i,x) - g(i,y). \end{aligned}$$

Now consider $i, j \in N$ and $x, y, z, v \in S$, such that $y \in [x]_i$, $z \in [y]_j$, $v \in [z]_i$ and $x \in [v]_j$. From the identity:

$$\frac{P_\delta(x)}{P_\delta(y)} \frac{P_\delta(y)}{P_\delta(z)} \frac{P_\delta(z)}{P_\delta(v)} \frac{P_\delta(v)}{P_\delta(x)} = 1,$$

[3] For a characterization on ordinal potential games, see Voorneveld and Norde [1997].

we derive that:

$$(g(i,x)-g(i,y))+(g(j,y)-g(j,z))+(g(i,z)-g(i,v))+(g(j,v)-g(j,x))=0,$$

which implies that the game $(N,S,\{g(i,.)\}_{i\in N})$ is a potential game (see Monderer and Shapley [1996], corollary 2.9). Hence, by definition, there exists a function $V:S\to\mathbb{R}$ such that for all $i\in N$, $x\in S$ and $y\in[x]_i$:

$$g(i,x)-g(i,y)=V(x)-V(y).$$

If G is an ordinal potential game with ordinal potential function V, it is easy to see that the perturbation $\Sigma_G(\delta)=\{p_\delta(x,y)\}_{x,y\in S}$ defined by $\frac{p_\delta(y,x)}{p_\delta(x,y)}=e^{\delta(V(x)-V(y))}$ is detailed balanced and utility monotonic. □

Applying theorem 1 allows us to deduce corollary 1 and 2. Corollary 1 shows that for ordinal potential games there is a one to one correspondence between the set of ordinal potential functions and the set of detailed balanced and utility monotonic perturbations:

Corollary 1. *For an ordinal potential game G, given a detailed balanced and utility monotonic perturbation $\Sigma_G(\delta)$ with utility monotonic function h we can construct an ordinal potential function V such that for all $i\in N, x\in S$ and $y\in[x]_i$:*

$$h(x,y)=V(x)-V(y);$$

Vice versa, for any ordinal potential function V of G, we can construct a utility monotonic and detailed balanced perturbation $\Sigma_G(\delta)$ with utility monotonic function h such that :$h(x,y)=V(x)-V(y)$.

Proof. The proof follows from the fact that the function $g(i,.)$ in the proof of theorem 1 is an ordinal potential function. □

A perturbation with utility monotonic function h is called a log linear perturbation if for all $i\in N$, $x\in S$ and $y\in[x]_i$: $h(x,y)=u_i(x)-u_i(y)$ (Blume [2003]). Then, corollary 2 states that this perturbation is detailed balanced if and only if the game is a potential game:

Corollary 2. *A game G has a detailed balanced and utility monotonic perturbation with utility monotonic function h that is given by:*

$$h(x,y)=u_i(x)-u_i(y)\quad\leftrightarrow\quad G \textit{ is a potential game.}$$

Proof. The proof follows directly from theorem 1. □

Obviously, ordinal potential games are a very interesting class of games when considering utility monotonic perturbations: they are the only class for which some of these perturbations also satisfy the detailed balance conditions. Moreover, the value of the potential function determines which states are stochastic stable and which are not.

The interesting result in corollary 2 has been noticed before (Baron et al. [2003]; Blume [2003]) and has been applied to various models, especially two by two symmetric population games. In these games, the state that maximizes the potential function, which is the stochastic stable state, results from the risk dominant strategy. This result is analogous to one obtained by using the uniform mutation model (Kandori et al. [1993]; Young [1993]). It therefore gave rise to the widespread view that the risk dominant strategies are robust to changes in the specification of the mutation rates.

However, we know that, in general, different ordinal potential functions have different maximal elements. Therefore, it is of interest to investigate which stochastic stable states are robust to variation of the ordinal potential function, or, equivalently, by corollary 1, variation of the perturbation. Theorem 2 establishes a rather discouraging result (the proof is provided in the appendix):

Theorem 2. *If G is an ordinal potential game and y is a stable state of M_G, then there exists a detailed balanced and utility monotonic perturbation of G such that y is a stochastic stable state of this perturbation.*

In other words, even within the class of ordinal potential games, the utility monotonicity condition is not strong enough to serve as a robust equilibrium refinement device: in order to generate robust equilibria we need a stronger utility monotonicity condition.

4 Conclusions

We show that ordinal potential games are characterized as the games which have detailed balanced and utility montonic perturbations. Therefore, we know that in more general stage games than the ordinal potential games, we cannot both satisfy detailed balancedness and utility monotonicity.

Furthermore, we show that within the class of ordinal potential games, utility monotonicity is not sufficient to generate robust equilibria; a stronger condition is required. Hence, in order to develop a universally applicable stochastic evolutionary equilibrium concept, we have to abandon the detailed balance condition; i.e. the stable distributions will have to be derived from non-reversible Markov chains.

While the Markov chain tree theorem provides an expression for the stable distributions in terms of *absolute* transition probabilities, it does not so in terms of *ratios* of transition probabilities, which is how the property of utility monotonicity is defined. We therefore believe that a further tightening of the bounds (Hordijk and Ridder [1988]) on the magnitude of stationary distributions, which are defined in terms of ratios of transition probabilities, may contribute a great deal to developing a (more) general equilibrium refinement device.

Appendix: Proof of theorem 2

The proof draws heavily on the theory of binary extensions. It is therefore of interest to briefly elucidate on some principles of this research field.

Consider a binary relation $R \subseteq S \times S$. The asymmetric part of R, denoted by $P(R)$ is defined by $(x, y) \in P(R)$ iff $(x, y) \in R$ and $(y, x) \notin R$. The symmetric part of R, denoted by $I(R)$ is given by $(x, y) \in I(R)$ iff $(x, y) \in R$ and $(y, x) \in R$. The transitive closure of R is denoted by $T(R)$ (cf. Section 2).

A binary relation R is reflexive if for all $x \in S$: $(x, x) \in R$; it is transitive if for all $x, y, z \in S$: $(x, y) \in R$ and $(y, z) \in R$ implies that $(x, z) \in R$; and R is complete if for all $x, y \in R$: $(x, y) \in R$ or $(y, x) \in R$. A reflexive and transitive relation is called a quasi-ordering and a complete quasi-ordering is called an ordering.

For a binary relation R the set of maximal elements of R, denoted by $U(R)$, is given by $U(R) = \{x \in S \mid \text{there is no } y \in S : (y, x) \in P(R)\}$.

The set of greatest elements of a binary relation R is denoted by $G(R)$ and is defined as: $G(R) = \{x \in S \mid \text{for all } y \in S : (x, y) \in R\}$.

A binary relation R' is said to be an extension of R if $R \subseteq R'$ and $P(R) \subseteq P(R')$. Suzumura [1976] derives the following relation:

Lemma 1. *A binary relation R has an ordering extension if and only if for all $x, y \in S$: $(x, y) \in T(R)$ implies $(y, x) \notin P(R)$.*

Furthermore:

Lemma 2. *If for all $y, x \in S : (x, y) \in T(R) \to (y, x) \notin P(R)$, then every ordering extension of R is also an ordering extension of $T(R)$.*

Proof. Let R' be an extension of R. If $(x, y) \in T(R)$, then, from the requirement that R' is transitive: $(x, y) \in R'$. Now assume on the contrary that $(x, y) \in P(T(R))$ and $(x, y) \notin P(R')$. From completeness of R' we have that $(y, x) \in R$. From $(x, y) \in T(R)$, we have that there is a sequence $x_1, ..., x_n$ such that $x_1 = x$, $x_n = y$ and for all $t = 1, .., n-1$: $(x_t, x_{t+1}) \in R$. There must also be a t such that $(x_t, x_{t+1}) \in P(R)$, otherwise we would have that $(y, x) \in T(R)$, which contradicts with $(x, y) \in P(T(R))$. From transitivity of R', we have that $(x_{t+1}, x_t) \in R'$ which, in turn, contradicts with $(x_t, x_{t+1}) \in P(R)$ and the requirement that R' is an extension of R. □

Every quasi-ordering satisfies the requirement of lemma 2, and has therefore an ordering extension. For a quasi-ordering R, let $\mathcal{E}(R)$ be the nonempty set of ordering extensions of R. The following result is due to Banerjee and Pattanaik [1996]:

Lemma 3. *The set of maximal elements of a quasi-ordering is equal to the union of the sets of greatest elements of its ordering extensions. Or, formally, for a quasi-ordering R:*

$$U(R) = \bigcup_{R' \in \mathcal{E}(R)} G(R').$$

Consider an ordinal potential game $G = \left(N, S, \{u_i\}_{i \in N}\right)$. The better than relation B is defined by:

$$(x, y) \in B \text{ if and only if } \exists i \in N : y \in [x]_i \text{ and } u_i(x) \geq u_i(y).$$

Voorneveld and Norde [1997] derive the following result:

Lemma 4. *A game with finite strategy set is an ordinal potential game if and only if for all $x, y \in S$: $(x,y) \in T(B)$ implies $(y,x) \notin P(B)$.*

Together with lemma 1 we derive that a game is an ordinal potential game if and only if the better than relation B has an ordering extension.

Consider an ordering extension R of B and define a real valued function $V : S \to \mathbb{R}$ such that $(x,y) \in R$ if and only if $V(x) \geq V(y)$. Since the set of strategy profiles, S, is finite, such a function can always be found. It is easy to see that the function V is an ordinal potential function of the game G. In a similar way every ordinal potential function of the game G defines an ordering extension of the relation B. The set of elements of S which maximizes the ordinal potential functions of G is then equal to the set:

$$\bigcup_{R' \in \mathcal{E}(B)} G(R') = \bigcup_{R' \in \mathcal{E}(T(B))} G(R') = U(T(B)).$$

The first equality stems from lemma 2, the second from lemma 3 and the fact that B is reflexive. That is, the set of elements which maximizes a potential of G is equal to the maximal elements of the relation $T(B)$. To complete the proof, we only have to show that $U(T(B))$ is equal to the union of the stable sets of the Markov chain M_G.

Consider the relation R_{M_G}. We can easy establish that the union of the stable sets of the Markov chain M_G is equal to $U(T(R_{M_G}))$. We have the following results:

Lemma 5. *$P(T(R_{M_G})) \subseteq P(T(B))$.*

Proof. From $R_{M_G} \subseteq B$, we see that $T(R_{M_G}) \subseteq T(B)$. Assume on the contrary that $(x,y) \in P(T(R_{M_G}))$ and $(y,x) \in T(B)$. Then there exists a sequence $x = x_1, ..., x_n = y$ such that for all $t = 1, ..., n-1$: $(x_t, x_{t+1}) \in R_{M_G}$ and for at least one $i \leq n-1$, $(x_i, x_{i+1}) \in P(R_{M_G})$. As $P(R_{M_G}) \subseteq P(B)$, we derive that $(x_i, x_{i+1}) \in P(B)$ and from transitivity of $T(B)$ we derive that $(x_{i+1}, x_i) \in T(B)$, which contradicts with lemma 4. □

Lemma 6. *$I(T(R_{M_G})) \subseteq I(T(B))$.*

Proof. Follows immediately from $T(R_{M_G}) \subseteq T(B)$, which itself is a consequence of $R_{M_G} \subseteq B$. □

Lemma 7. *$U(T(R_{M_G})) \subseteq U(T(B))$.*

Proof. Let $x \in U(T(R_{M_G}))$ and assme that there is an $y \in S : (y,x) \in P(T(B))$. Then there is a sequence $y = x_1, ..., x_n = x$ such that for all $t = 1, ..., n-1$:$(x_t, x_{t+1}) \in B$. As $(x_{n-1}, x) \in B$ and x is maximal, we must have that $(x_{n-1}, x) \in I(R_{M_G})$, hence x_{n-1} is also maximal. By iteration, we must have that y is also maximal, implying that $(x,y) \in I(T(R_{M_G}))$. From lemma 6, we derive that $(x,y) \in I(T(B))$, a contradiction. □

Lemma 8. *$U(T(B)) \subseteq U(T(R_{M_G}))$.*

Proof. Let $x \in U(T(B))$ and assume on the contrary that there is an $y \in S : (y,x) \in P(T(R_{M_G}))$. From lemma 5, we derive that $(y,x) \in P(T(B))$, which contradicts with maximality of x. □

The desired result follows from combining lemma 7 and 8.

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