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WORKING PAPER

Existence of closed and complete extensions

applied to convex, homothetic an monotonic orderings

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Many theories of consumer demand impose specific properties on the preference relations, e.g. convexity, monotonicity or homotheticity. Existing non-parametric tests which allow us to single out the preference relations that do not satisfy these properties are only valid in very specific contexts. This paper is an attempt to address this lacuna in the literature. We provide a theorem on the existence of complete binary extensions that satisfy properties which are closed under intersection. From this theorem we derive necessary and sufficient conditions for the existence of convex, homothetic and monotonic orderings on general domains.

1 Introduction

Many theories of consumer demand impose specific properties on the preference relations, like convexity, monotonicity and homotheticity. As these properties have strong implications for the results of the models for which they are used, it is important to know whether actual preferences satisfy them. In reality, however, instead of observing preferences, one has observations on choices. By noticing that the chosen alternatives from a set are the most preferred alternatives in this set, we may test restrictions on the preference relation via tests on the observed choices. Deriving such test procedures, the so called rationalizability problem, is the subject of revealed preference theory.

Existing revealed preference tests which allow to single out the preference relations that do not satisfy above mentioned properties are only valid in very specific contexts. Varian (1983) develops tests to verify for the properties of convexity and homotheticity. These results assume that the opportunity sets take the form of budget sets. Liu and Wong (2000) extend these results but keep the assumption on the form of the opportunity sets. Richter and Wong (2004) develop a test for the existence of convex preferences, but they assume that the observed preferences are finite, complete and transitive. Richter (1966) provides a condition that characterizes the existence of a transitive and complete preference relation on general domains. His condition, however, says nothing about the existence of convex, monotonic or homothetic preference orderings. To our knowledge, the only research that provides a characterization of (strict) convex and monotonic preferences on general domains is Bossert and Sprumont (2001). Their result applies to a framework of non-deteriorating choice, but it is easy to see that it can also be applied to the revealed preference framework under the condition that the revealed preference relation is finite. Besides the assumption that the observed relation is finite, their work distinguishes from ours by using different concepts and different methods of proof, and consequentially can not be directly compared.

Consider following simple example: an individual has a preference relation over a convex subset of 3-dimensional real space. Consider six consumption bundles in this set:

$$\begin{aligned} x_1 &= (10,9,3) \\ x_2 &= (9,7,8) \\ x_3 &= (12,4,8) \\ x_4 &= (4,8,12) \\ x_5 &= (8,9,7) \\ x_6 &= (8,12,4) \end{aligned}$$

Assume that we observe that x_1 is preferred to x_2 and x_5, x_3 is preferred to x_4, x_6 is preferred to x_4 and x_4 is strictly preferred to x_1. Is it possible that the preference relation of this individual is a convex ordering? Current non-parametric tests are unable to answer this question since the required domain conditions are not met[1]. In section 3, we will reconsider this example and argue that the individual preferences are not convex.

The use of general domains requires a general approach. For instance, it implies that a priori there are no restrictions on the observed revealed preference relation. In this perspective, we may drop the concept of choice function and work directly on the revealed preference relation, which can be any relation defined on the set of alternatives. This view leads us to the theory of binary extensions: an extension of a (revealed preference) relation is a binary relation that conserves the symmetric and asymmetric parts. The true preference relation is now viewed as a binary extension of the revealed preference relation and the demand for rationalizability is replaced by the demand for the existence of a binary extension satisfying a certain list of properties. The theory on binary extensions was initiated by Szpilrajn (1930) and was consecutively elaborated by, among others, Dushnik and Miller (1941), Suzumura (1976), Donaldson and Weymark (1998) and Duggan (1999). This research topic focussed almost entirely on the properties of transitivity and completeness[2].

By developing characterizations for the existence of binary extensions which satisfy –besides transitivity and completeness– the properties of convexity, homotheticity and monotonicity on general domains, our paper meets the call by Bossert et al. (2002) for a characterization of ordering extensions which satisfy properties such as convexity or monotonicity.

The main results of this paper are summarized below.

A property k of a binary relation is closed under intersection if the intersection of any set of relations satisfying k also satisfies k. Examples are transitivity, convexity, monotonicity and homotheticity. The set of properties which are closed under intersection is in a one to one relationship with the set of closure operators. A closure operator is an increasing, monotonic and idempotent function from the set of binary relations to itself. The set of relations which satisfy property k is then equal to the set of fixed points of the associated closure operator K. For a given relation R, the closure of R, $K(R)$, is the smallest relation which includes R and satisfies property k. This leads to the following, almost tautological,

[1] The opportunity sets do not have the form of budget sets as required by Varian (1983) and the observed preferences are not complete (e.g. x_1 vs x_3) nor transitive as is required in by Richter and Wong (2004)

[2] Duggan (1999) is an exception. On the other hand, his properties have no direct relevance to preference relations.

result on the existence of binary extensions that satisfy property k: a binary relation has an extension which satisfies k if the closure of R, $K(R)$, does not conflict with its asymmetric part.

So far, everything works out very nice. However, there is a catch: the property of completeness is not closed under intersection and has therefore, no representation as a closure operator. In order to solve this issue, we impose restrictions on the closure operator such that the characterization result holds even if we impose the additional property of completeness on the binary extension. A first restriction requires the closure operator to be algebraic. This means that the closure of a binary relation can be recovered from the closures of its finite subsets. The second restriction is called non-comparable extendibility and requires that for a closed relation, the closure of the union of this relation with an element of its non-comparable part is an extension of this closed relation. Our result then reads: if property k is closed under intersection and its associated closure operator K is algebraic and non-comparable extendible, then a binary relation R has a complete extension that satisfies property k if and only if its closure $K(R)$ does not conflict with its asymmetric part. For each of the properties of transitivity, convexity, monotonicity and homotheticity, we define suitable closure operators and we show that they are algebraic and non-comparable extendible. As such, we are able to characterize the existence of complete and transitive relations which satisfies convexity, monotonicity or homotheticity. Further, we state and prove the corresponding rationalizability results.

In section 2 we introduce notation and basic definitions and derive the main theorem of the paper. It also presents a generalization of the result of Donaldson and Weymark (1998) that every transitive and reflexive relation can be recovered from its ordering extensions. Section 3 applies the main theorem to specific closure operators, in which we focus on the properties of convexity, homotheticity and monotonicity. In section 4 we build a bridge between the revealed preference literature and our results. This allows us to construct non-parametric tests that can be taken to the data. Section 5 presents conclusions.

2 The general extension result

As usual $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_0^+$ is the set of all strict positive real numbers and $\mathbb{R}^n$ is the n-fold cartesian product of $\mathbb{R}$. The set $\mathbb{N}$ is the set of strict positive integers. For two elements x and y in $\mathbb{R}^n$, we write $x \geq y$ if every element of x is larger or equal than every corresponding element in y and we write $x > y$ if $x \geq y$ and $x \neq y$. For a universal set X, we say that R is a binary relation on X if R is a subset of $X \times X$. The inverse relation of R, denoted by R^{-1} is defined as: $(x, y) \in R^{-1}$ if and only if $(x, y) \notin R$. The *symmetric part* $R \cap R^{-1}$ is denoted by $I(R)$, the *asymmetric part* $R - I(R)$ by $P(R)$ and the *non-comparable part* $X \times X - (R \cup R^{-1})$ by $N(R)$.

A binary relation, R, is *reflexive* if for all x in X, $(x, x) \in R$, it is *transitive* if for all x, y and z in X, $(x, y) \in R$ and $(y, z) \in R$, implies $(x, z) \in R$ and it is *complete* if for all x and

y in X either $(x, y) \in R$ or $(y, x) \in R$. Notice that a complete relation is always reflexive. A transitive and complete relation is called an *ordering*.

The set of binary relations on X is given by $\mathcal{R} = \{R \,|R \subseteq X \times X\}$.

Definition 1 *A property of a binary relation is a function k from the set $\mathcal{R}$ to the set $\{0, 1\}$.*

Given a binary relation $R \in \mathcal{R}$, we say that R satisfies the property k if and only if $k(R) = 1$. On the other hand, R violates k if and only if $k(R) = 0$.

Definition 2 *A property k is* closed under intersection *if for an index set A and for all $i \in A$: $k(R_i) = 1$ implies that*

$$k\left(\bigcap_{i \in A} R_i\right) = 1.$$

Evaluating the properties mentioned above, we see that transitivity and reflexivity are both closed under intersection. On the other hand, next example shows that completeness is not closed under intersection.

Example 1.
Let X be any subset of $\mathbb{R}$, $R = \{(x, y) \in X \times X \,|x \leq y\}$ and let $Q = R^{-1}$. Then both R and Q are complete bur $R \cap Q = \emptyset$ is not.

Definition 3 *A closure operator K is a function from $\mathcal{R}$ to $\mathcal{R}$ for which:*

1. *$R \subseteq K(R)$,*
2. *if $R \subseteq R'$, then $K(R) \subseteq K(R')$,*
3. *$K(K(R)) = K(R)$,*

The first property states that the closure operator is increasing, the second property states that it is monotonic and the third property states that it is idempotent.

Following Lemma (which we proof for completeness) shows that there is a one to one mapping between the set of properties which are closed under intersection and the set of closure operators.

Lemma 1 *Given a property k which is closed under intersection, the function $K : \mathcal{R} \to \mathcal{R}$ defined by:*

$$K(R) = \bigcap \{Q \supseteq R \,|k(Q) = 1\}$$

is a closure operator

Given a closure operator K, the function $k : \mathcal{R} \to \{0, 1\}$ defined by:

$$k(R) = 1 \leftrightarrow K(R) = R$$

is a property which is closed under intersection.

Proof: Let k be a property which is closed under intersection. It is easy to verify that the first and second property of Definition 3 are satisfied. To show that K is idempotent, notice that, as k is closed under intersection, for all R: $k(K(R)) = 1$. As such, $K(R)$ satisfies property k. Further for every R with $k(R) = 1$, $K(R) = R$. Therefore $K(K(R)) = K(R)$ for all $R \in \mathcal{R}$.

Given a closure operator K, let us show that k is closed under intersection. Consider an index set A and for all $i \in A$ a relation $R_i = K(R_i)$. As for each $i \in A$: $\bigcap_{j \in A} R_j \subseteq R_i$, we have by property 2 of the closure operator: $K\left(\bigcap_{j \in A} R_j\right) \subseteq \bigcap_{j \in A} K(R_j)$. From the first property of the closure operator, we have that $\bigcap_{j \in A} R_j \subseteq K\left(\bigcap_{j \in A} R_j\right)$. This implies that $K\left(\bigcap_{j \in A} R_j\right) = \bigcap_{j \in A} R_j$. $\square$

From now on properties will be denoted by small letters (e.g. k, t, h) and the corresponding closure operators will be denoted by capital letters (e.g. K, T, H).

In theory it should be easy to verify whether a binary relation (preference relation) satisfies a certain list of properties. To often, however, one observes only a subset of the true preference relation. In this case, it is no longer possible to verify the properties directly from the true preferences. At best we can show the existence of a binary relation with the required properties and state that it is possible for the true preference relation to be equal to this relation.

Definition 4 *A binary relation R^* is an extension of R if and only if $R \subseteq R^*$ and $P(R) \subseteq P(R^*)$.*

We identify R^* with the true preference relation and we identify R with the observed preference relation. The requirement for the existence a preference relation satisfying certain properties is now translated to the requirement of the existence of a binary extension of R satisfying this list of properties. Let us first consider properties which are closed under intersection.

Definition 5 *Given a property k which is closed under intersection, a binary relation is k-consistent if its K-closure does not conflict with its asymmetric part, i.e.*

$$K(R) \cap P^{-1}(R) = \emptyset.$$

The next lemma gives the characterization for the existence of a binary extension which satisfies property k.

Lemma 2 *A relation R is k-consistent if and only if it has a binary extension satisfying property k.*

Proof. Let R be k-consistent. We show that $K(R)$, which satisfies property k by definition, is an extension of R. From the first property of the closure operator, $R \subseteq K(R)$. If on the contrary $(x,y) \in P(R)$ and $(x,y) \notin P(K(R))$, we have that $(y,x) \in K(R)$ and $(x,y) \in P(R)$, in contradiction with K-consistency of R.

Let R^* be an extension of R satisfying property k. If on the contrary R is not k-consistent, we have an element $(x,y) \in K(R)$ such that $(y,x) \in P(R)$. As R^* is an extension of R and from the definition of the K-closure we have that $K(R) \subseteq K(R^*) = R^*$. This implies that $(x,y) \in R^*$ and $(y,x) \in P(R^*)$, a contradiction. $\square$

As seen in Example 1, the property of completeness is not preserved under intersection. Therefore it can not be represented by a closure operator. On the other hand, completeness is a very essential property in economic theory and we would like to include it into the list of required properties. The remaining part of this section shows that for some closure operator, the additional requirement of completeness does not impose any new restrictions upon the observed preference relation.

Definition 6 *A closure operator K is algebraic if and only if for all $R \in \mathcal{R}$: $(x,y) \in K(R)$ implies that there is a finite subset of R, lets say R' such that $(x,y) = K(R')$.*

An algebraic closure of a binary relation can be recovered from the closures of its finite subsets. If the universal set X is finite, every closure operator is by definition algebraic. Therefore, algebraic closure operators only impose restrictions in case the domain is infinite.

Before we give the main result of this section, we impose one last restriction on the class of admissible closure operators.

Definition 7 *A closure operator K is non-comparable extendible if for all R in $\mathcal{R}$:*
$R = K(R)$ implies that for all $(x_0, y_0) \in N(R)$: $K(R \cup \{x_0, y_0\}) \cap P^{-1}(R \cup \{x_0, y_0\}) = \emptyset$.

At this point, we can state our main result.

Theorem 1 *Given a property k with can by represented by a non-comparable extendible, algebraic closure operator;*

a binary relation has a complete extension satisfying property k if and only if it is k-consistent

Proof. Let K be a non-comparable extendible algebraic closure operator representing property k and assume that R has a complete extension R^* such that $K(R^*) = R^*$. If on the contrary R is not k-consistent, there exist $x, y \in X$ such that $(x,y) \in K(R) \cap P^{-1}(R)$.

As $K(R) \subseteq K(R^*) = R^*$ and $P(R) \subseteq P(R^*)$, we obtain $(x, y) \in R^*$ and $(y, x) \in P(R^*)$, a contradiction. Conclude that R is k-consistent.

The reverse is a bit more involved. Let K be a non-comparable extendible, algebraic closure operator and assume that R is k-consistent. Consider the set Ω of all k-consistent extensions of R. This set is non-empty as it contains R. Consider a chain, Ω', in the set Ω, i.e. for all R' and R'' in Ω', either $R' \subseteq R''$ or $R'' \subseteq R'$. We will apply Zorn's lemma on the set Ω and show that all maximal elements of Ω are complete and closed.

Consider the relation $B = \bigcup_{R' \in \Omega'} R'$. As B is the union of extensions of R, it is also an extension of R. To show that B is k-consistent, assume on the contrary that $(x, y) \in K(B) \cap P^{-1}(B)$. The closure K is algebraic, so there must be a finite subset of B, lets say B', such that $(x, y) \in K(B')$ and by definition of B, there must be a relation R' in Ω' containing B'. Also, there is at least one relation R'' in Ω' such that $(y, x) \in P(R'')$ and for all relations R''' in Ω', $(x, y) \notin R'''$. If R' contains R'', we have that $(x, y) \in K(R')$ and $(y, x) \in P(R')$. If R'' contains R' we have that $(x, y) \in K(R'')$ and $(y, x) \in P(R'')$. In both cases, we have a contradiction with k-consistency of R' and R''. Conclude that B is k-consistent. Application of Zorn's lemma results in the existence of a maximal element in the set Ω.

Let R^* be a maximal element of Ω. We will show that R^* satisfies property k and completeness. To show the first, notice that $K(R^*)$ is an extension of R^*. As R^* is an extension of R, $K(R^*)$ is also an extension of R. Together with k-consistency of $K(R^*)$, we must conclude that $K(R^*) \in \Omega$. By the first property of the closure operator, $R^* \subseteq K(R^*)$ and by maximality of R^*, $K(R^*) = R^*$. Conclude that R^* satisfies property k. To show completeness of R^* assume on the contrary that there exist elements $x_0, y_0 \in X$ such that $(x_0, y_0) \in N(R^*)$. The relation $R_0 = R \cup \{x_0, y_0\}$ has by non-comparable extendability of K an extension $K(R_0)$. This relation is also an extension of R^* and as such, also an extension of R. Furthermore, $K(R_0)$ is k-consistent, hence it is in the set Ω. As $K(R_0)$ strictly contains R^*, we have a contradiction with maximality of R^*. Conclude that R^* is complete. $\square$

A relation satisfying k is also k-consistent. Hence, by Theorem 1, it has a complete extension satisfying property k. Let $\mathcal{E}(R)$ be the non-empty set of all complete extensions satisfying property k of a relation $R = K(R)$ (i.e. R also satisfies property k). The following corollary gives a result similar to the result of Donaldson and Weymark (1998).

Corollary 1 *If a property k corresponds to a closure operator K which is algebraic and non-comparable extendible, then for any reflexive relation R satisfying property k:*

$$R = \bigcap_{R' \in \mathcal{E}(R)} R'.$$

Proof. The relation R is contained in every element of the set $\mathcal{E}(R)$, hence it is also in their intersection.

To see the converse, assume on the contrary that (x, y) is contained in every element $\mathcal{E}(R)$ and that $(x, y) \notin R$. If $(y, x) \in P(R)$, we have a contradiction with the assumption that $\mathcal{E}(R)$ contains the extensions of R, hence we must have that $(x, y) \in N(R)$. The relation $R_0 = R \cup \{y, x\}$ has by non-comparable extendibility of K, as extension the relation $K(R_0)$. This relation is also an extension of R. As $K(R_0)$ is k-consistent it has by Theorem 1 a complete extension satisfying property k, lets say R^*. The relation R^* is also an extension of R, hence it is in $\mathcal{E}(R)$. The contradiction follows from the fact that $(y, x) \in P(R^*)$. $\square$

3 Transitive, convex, homothetic and monotonic extensions

This section applies Theorem 1 to specific properties. We focus on the properties of transitivity, convexity, homotheticity and monotonicity, as these are, in our opinion, the most frequently imposed properties on binary relations in economic theory.

3.1 Complete and transitive extensions

To start, we reproduce the result of Suzumura (1976) that every relation has a complet and transitive extension if and only if it is transitive-consistent. Additionally, we obtain the result of Donaldson and Weymark (1998) that every reflexive and transitive relation equals the intersection of its ordering extensions.

Let R be a binary relation on our universal space X. Recall the definition of transitivity:

Definition 8 *The relation R is transitive (property t) if for all x, y and z in X,*

$$(x, y) \in R \text{ and } (y, z) \in R \text{ implies that } (x, z) \in R.$$

The *transitive closure* of R is denoted by $T(R)$ and is defined in the following way:

Definition 9 $(x, y) \in T(R)$ *if $x = y$ or there is a sequence $s = x_1, x_2, ..., x_n$ of elements in X such that $x = x_1$, $x_n = y$ and for each $i = 1, ..., n-1$:*

$$(x_i, x_{i+1}) \in R.$$

Let us start by showing that indeed T is the closure operator corresponding to property t.

Lemma 3

$$T(R) = \bigcap \{Q \supseteq R \,| t(Q) = 1\}$$

Proof: Assume that $(x, y) \in T(R)$ and there is a transitive relation $Q \supseteq R$. From the definition, we have a sequence $x_1, ..., x_n$ such that $x_1 = x$, $x_n = y$ and for all $i = 1, ..., n-1$: $(x_i, x_{i+1}) \in R$. From $R \subseteq Q$, we have that $(x_i, x_{i+1}) \in Q$ for all $i = 1, ..., n$. By repeated application of transitivity we have that $(x, y) \in Q$.

By Definition 3: $R \subseteq T(R)$. It is also easy to see that $T(R)$ is transitive. Therefore we have that when $(x, y) \in Q$ for all transitive relations containing R, $(x, y) \in T(R)$. $\square$

In order to apply Theorem 1 to the transitive closure, we need to show that it is algebraic and non-comparable extendible. We do this in following lemma.

Lemma 4 *The closure T is algebraic and non-comparable extendible.*

Proof. To see that T is algebraic, let $(x, y) \in T(R)$. Then, either $x = y$ or there is a number $n \in \mathbb{N}$ and a sequence $s = x_1, ..., x_n$ of elements in X such that for all $i = 1, ..., n-1$:

$$(x_i, x_{i+1}) \in R.$$

Let the set A collect all elements of the sequence s. By finiteness of the sequence s, $R \cap A \times A$ is a finite subset of R. The conclusion follows from the observation that $(x, y) \in T(R \cap A \times A)$. $\square$

Let us now show that T is non-comparable extendible. Let $R = T(R)$, $(x_0, y_0) \in N(R)$ and $R_0 = R \cup \{(x_0, y_0)\}$. From Definition 3, we have that $R_0 \subseteq T(R_0)$. Assume on the contrary that $(x, y) \in T(R_0) \cap P^{-1}(R)$.

From $(x, y) \in T(R_0)$, we know that there is a number $n \in \mathbb{N}$ and a sequence $s = x_1, ..., x_n$ such that $x = x_1$, $y = x_n$ and for all $i = 1, 2, ...n-1$: $(x_i, x_{i+1}) \in R_0$. If for all $i = 1, 2, ..., n-1$ also $(x_i, x_{i+1}) \in R$, we have that $(x, y) \in T(R)$ a contradiction with the assumption that $(y, x) \in P(R_0)$.

We conclude that there must at least be one $i \leq n$ such that $(x_i, x_{i+1}) = (x_0, y_0)$. Let x_l be the last instance of such an y_0 and let x_f be the first instance of such an x_0 in the sequence s. As $R_0 - R = (x_0, y_0)$ and $(y, x) \in R_0$ we have that $(x_l, x_f) \in T(R)$, or equivalently $(y_0, x_0) \in R$, a contradiction. $\square$

The following theorem reproduces the result of Suzumura (1976).

Theorem 2 *R has an ordering (transitive and complete) extension if and only if R is t-consistent, i.e. $T(R) \cap P^{-1}(R) = \emptyset$.*

Proof. By Lemma 4, the closure T closure is algebraic and non-comparable extendible. Further, by Lemma 3, T, is the closure operator corresponding to property t. Application of Theorem 1 implies that a relation has a complete and transitive extension if and only if it is t-consistent. $\square$

As additional result, we can reproduce the result of Donaldson and Weymark (1989).

Theorem 3 *A reflexive and transitive relation is equal to the intersection of its ordering extensions.*

Proof. Follows immediately from Theorem 1, 2 and Corollary 1. □

3.2 Complete, transitive and convex extensions

In this part, we focus on the additional property of convexity. For this we take our universal space to be a subset of $\mathbb{R}^m$. Further, we assume that X is convex closed, i.e. every convex combination of two elements of X is in X. It is possible to reproduce the results of this section without this domain assumption. However, this would drastically expand the notational difficulty without really adding something fundamental to the analysis.

The property of convexity has many forms, depending on the additional requirements imposed on the relation under consideration. In its most fundamental form, it may be stated as follow:

Definition 10 *A relation R is convex if*

$$(x, y) \in R \text{ (resp. } P(R)) \text{ and } 0 < \alpha < 1 \text{ implies } (\alpha x + (1-\alpha) y, y) \in R \text{ (resp. } P(R)).$$

Four our purpose this definition is not really adequate so we define our property in another way. Consider a finite set $A \subseteq X$ and let

$$V(A) = \left\{ x \in X \middle| x = \sum_{y_i \in A} \alpha_i y_i \right\}$$

where for all i, $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. The set $V(A)$ is the convex hull spanned by the elements of A without its boundaries.

Consider a finite number $n \in \mathbb{N}$ of sequences $s^1, ..., s^n$ of finite length. We denote the jth element of sequence i by x_j^i. The closure operator C of a relation R, denoted by $C(R)$ is defined as:

Definition 11 *$(x, y) \in C(R)$ if $x = y$ or there is a number $n \in \mathbb{N}$ of sequences $s^1, ..., s^n$ of finite length, where:*

- *all initial values of all sequences are equal to x (i.e. for all $i = 1, 2, ..., n$, $x_1^i = x$),*
- *all terminal values of all sequences are equal to y (i.e. for the sequence s^i of length n_i: $x_{n_i}^i = y$),*
- *all nonterminal elements x_j^i of a sequence s^i ($i = 1, 2, ..., n$), are related to their immediate successor in the sequence s^i, i.e. x_{j+1}^i, by one of following two rules*

1. $\left(x_j^i, x_{j+1}^i\right) \in R$ *or*
2. $x_j^i \in V\left(A_j^i\right)$, *where* A_j^i *is a subset of the set which collects all the elements of the sequences* $s^1, ..., s^n$ *and* A_j^i *contains* x_{j+1}^i.

We remark that although two elements can be in a different sequence or in a different position of the same sequence, it is possible that they represent the same element in the set X.

In this section we define the property c by its corresponding closure C. This approach is equivalent to the approach form previous section, as we can define the property c as $c(R) = 1$ if and only if $C(R) = R$.

It is not immediately clear how this property c corresponds to the notion of convexity and transitivity. However, we have following Lemma.

Lemma 5 *If R is complete, then R is convex and transitive if and only if $R = C(R)$ (i.e. R satisfies property c).*

Proof. It is easy to see that $R = C(R)$ implies that R is transitive and convex. The converse is a bit more complicated. We proceed by showing:

(i) If R is complete, transitive and convex, then for every finite subset A of X if

$$y \in V(A)$$

we have that either $(y, y_j) \in I(R)$ for all $y_j \in A$ or $(y, y_j) \in P(R)$ for at least one $y_j \in A$.

(ii) If R is transitive and satisfies the condition under (i), then $R = C(R)$.

To proof (i), let R be complete, transitive and convex. Let $y \in V(A)$. We must show that either $(y, y_j) \in I(R)$ for all $j = 1, 2, ..., n$ or for at least one $j = 1, 2, ..., n$, $(y, y_j) \in P(R)$.

The proof is by induction on n. If $n = 1$, then we have $y = y$, and by completeness of R : $(y, y) \in I(R)$. Let (i) be satisfied for all $n \leq m$. Now let $y = \sum_{j=1}^{m+1} \alpha_j y_j$. Consider $y' = \sum_{j=1}^{m} \frac{\alpha_j}{1-\alpha_{m+1}} y_j$. This element y' is in X, because X is a convex set. We have that

$$y = \alpha_{m+1} y_{m+1} + (1 - \alpha_{m+1}) y'.$$

By the induction hypothesis either $(y', y_j) \in I(R)$ for all $j \leq m$ or $(y', y_j) \in P(R)$ for at least one $j \leq m$.

The relation R is complete, hence either $(y_{m+1}, y') \in R$ or $(y', y_{m+1}) \in R$.

If $(y_{m+1}, y') \in R$, we have by convexity that $(y, y') \in R$. If $(y, y') \in P(R)$, we have, together with the induction hypothesis and transitivity, that $(y, y_j) \in P(R)$ for at least one $j \leq m$. If $(y, y') \in I(R)$, by convexity also $(y, y_{m+1}) \in I(R)$. By the induction hypothesis and transitivity either $(y, y_j) \in P(R)$ for at least one $j \leq m$ or $(y, y_j) \in I(R)$ for all $j \leq m+1$.

If $(y', y_{m+1}) \in R$ we have by convexity that $(y, y_{m+1}) \in R$. If $(y, y_{m+1}) \in P(R)$, we are done. If $(y_{m+1}, y) \in I(R)$ we also have by convexity $(y_{m+1}, y') \in I(R)$. From the induction hypothesis and transitivity, either $(y, y_j) \in P(R)$ for at least one $j \leq m$ or $(y, y_j) \in I(R)$ for all $j \leq m+1$.

(ii) Let R be transitive, convex and complete. That $R \subseteq C(R)$ follows immediately from the definition of the convex closure. Assume that $(x, y) \in C(R)$. We show that $(x, y) \in R$ via the construction of a sequence $y_1, ..., y_m$ such that $y_1 = x$, $y_m = y$ and for all $j = 1, 2, ..., m-1$, $(y_j, y_{j+1}) \in R$. Consider following algorithm:

1. Put $y_1 = x_1^1$ and set $m = 1$.
2. If $y_m = y$, we stop. Otherwise, we increase m by one $(m := m+1)$,
3. For $y_{m-1} = x_j^i$, if $\left(x_j^i, x_{j+1}^i\right) \in R$, we put y_m equal to x_{j+1}^i and return to step 2.
4. For $y_{m-1} = x_j^i$, if $x_j^i \in V\left(A_j^i\right)$, by step (i), there are two cases to consider.
   (a) If $\left(x_j^i, x_{j+1}^i\right) \in R$, we put $y_m = x_{j+1}^i$ and return to step 2.
   (b) If $\left(x_j^i, x_w^v\right) \in P(R)$ for some element x_w^v in a sequence s^v, we put $y_m = x_w^v$ and return to step 2.

To get an idea how this algorithm works, assume that we have arrived at an element x_j^i. First the algorithm looks at the element x_{j+1}^i. If $\left(x_j^i, x_{j+1}^i\right) \in R$, the algorithm considers x_{j+1}^i as the following element. If $x_j^i \in V\left(A_j^i\right)$, by step (i) of the proof, there are two cases. Either $\left(x_j^i, x_{j+1}^i\right) \in R$, in which case the algorithm considers x_{j+1}^i as the following element or there is an element x_w^v in a sequence s^v, such that $\left(x_j^i, x_w^v\right) \in P(R)$. In this case, the algorithm considers x_w^v as the following element.

The algorithm is well behaved, because it stops at the value y. We show that the algorithm stops in finite time. If it does not, by finiteness of the sequences, we must have a loop in the sequence $x = y_1, y_2, ..., y_f, ..., y_l, ...$ Lets say y_f and y_l correspond to the same element in the same sequence. This can only occur if the algorithm suddenly jumps from one sequence to another, i.e. it passes step 4.(b). Therefore, there must be a strict preference involved, lets say $(y_v, y_{v+1}) \in P(R)$ $(f \leq v \leq l)$. Also, as y_v is in the loop,we must have that $(y_{v+1}, y_v) \in T(R)$. This contradicts with the assumption that R is transitive. Therefore, the algorithm must stop in finite time at step 2, i.e. at y. By transitivity of R, we get $(x, y) \in R$, as desired. □

In order to use Theorem 1, we still need to show that C is an algebraic closure operator and that it is non-comparable extendible. This is shown in the following lemmata.

Lemma 6 *The operator T is an algebraic closure operator.*

Proof. Property 1 and 2 of the closure operator are easily verified. To see property 3, let $(x,y) \in C(C(R))$. By definition either $x = y$ or there exist a number $n \in \mathbb{N}$ and sequences $s^1, ..., s^n$ of finite length where all sequences start with x, end with y and for each nonterminal element x_j^i in a sequence s^i and its immediate successor x_{j+1}^i either $\left(x_j^i, x_{j+1}^i\right) \in C(R)$ or $x_j^i \in V\left(A_j^i\right)$, where A_j^i is a set containing x_{j+1}^i and a number of elements of all the sequences.

If $x = y$, we immediately have that $(x,y) \in C(R)$. If $x \neq y$, we see that for each jth element in a sequence s^i, i.e. x_j^i, for which $\left(x_j^i, x_{j+1}^i\right) \in C(R)$ there is a finite number, $n(i,j) \in \mathbb{N}$, of sequences $s(i,j)^1, ..., s(i,j)^{n(i,j)}$, such that all sequences start with x_j^i, end with x_{j+1}^i and for each nonterminal element $x(i,j)_w^v$ in a sequence $s(i,j)^v$, either $\left(x(i,j)_w^v, x(i,j)_{w+1}^v\right) \in R$ or $x(i,j)_w^v \in V\left(A(i,j)_w^v\right)$, where $A(i,j)_w^v$ contains $x(i,j)_{w+1}^v$ and a number of elements of the sequences $s(i,j)^1, ..., s(i,j)^{n(i,j)}$.

For an element x_j^i in sequence s^i for which $x_j^i \in V\left(A_j^i\right)$, we consider the sequence $s(i,j)^1 = x(i,j)_1^1, x(i,j)_2^1$, where the first element equals x_j^i and the second element equals x_{j+1}^i. Now, let $s'(i,j)^v$ be the sequence $s(i,j)^v$ without the last element. Let sequence s^i have n_i elements. Construct the (finite number) of sequences of the form

$$s(i,j,v) = s'(i,1)^1, s'(i,2)^1, ..., s'(i,j-1)^1, s'(i,j)^v, s'(i,j+1)^1, ..., s(i,n_i-1)^1$$

All these sequences, start with the value x and end with the value y. Further, each nonterminal element in this sequence and its immediate successor is linked in a way such that $(x,y) \in C(R)$.

To show that C is algebraic, let $(x,y) \in C(R)$. We have a finite number of sequences $s^1, ..., s^n$ where each sequence begins with x, ends with y and for each nonterminal element x_j^i in a sequence s^i and its immediate successor x_{j+1}^i, either $\left(x_j^i, x_{j+1}^i\right) \in R$ or $x_j^i \in V\left(A_j^i\right)$. Let A collect all the elements of the sequences $s^1, ..., s^n$.The proof follows from the fact that $(x,y) \in C(R \cap A \times A)$ and that $R \cap A \times A$ is a finite subset of R. $\square$

Lemma 7 *The closure C is non-comparable extendible.*

Proof. Let $R = C(R)$ and $(x_0, y_0) \in N(R)$. As $C(R)$ is reflexive $x \neq y$. Assume on the contrary that for $R_0 = R \cup \{(x_0, y_0)\}$: $(x,y) \in C(R_0) \cap P^{-1}(R_0)$. There are two cases to consider.

1. $(y,x) \in P(R)$.

   From the definition of C, there exist sequences $s^1, ..., s^n$ where each sequence starts with x, ends with y and for each nonterminal element x_j^i in a sequence s^i and its immediate successor x_{j+1}^i either $\left(x_j^i, x_{j+1}^i\right) \in R_0$ or $x_j^i \in V\left(A_j^i\right)$, where A_j^i contains x_{j+1}^i and a number of elements from the sequences $s^1, ..., s^n$.

If for all x_j^i, where $\left(x_j^i, x_{j+1}^i\right) \in R_0$ also $\left(x_j^i, x_{j+1}^i\right) \in R$, then $(x, y) \in C(R)$, a contradiction. Therefore, there is at least one x_j^i for which $x_j^i \notin V\left(A_j^i\right)$ and $\left(x_j^i, x_{j+1}^i\right) = (x_0, y_0)$.

Take any sequence s^i and denote its length by n_i. If there is an element x_j^i in s^i so that $\left(x_j^i, x_{j+1}^i\right) = (x_0, y_0)$, there are at most a finite number of such elements. Let $l-1$ be the last such value of j and let f be the first such value of j (we have $x_l^i = y_0$ and $x_f^i = x_0$). For each instance of j for which $\left(x_j^i, x_{j+1}^i\right) = (x_0, y_0)$ either $j+1 = l$ in which case we consider the sequence

$$y_0 = x_{l+1}^i, ...x_{n_i-1}^i, y, x, x_2^i, ..., x_f^i = x_0$$

...or there must be a value of x_0 further in the sequence (i.e. there is an element x_v^i in sequence s^i, with $v > j$ so that $\left(x_v^i, x_{v+1}^i\right) = (x_0, y_0)$). In this case, we take the sequence

$$y_0 = x_{j+1}^i, ..., x_v^i = x_0.$$

If in the sequence s^i, there is no element x_j^i for which $\left(x_j^i, x_{j+1}^i\right) = (x_0, y_0)$, we consider the sequence

$$x_0 = x_l^v, ..., x_{n_v-1}^v, y, x, x_2^i, ..., x_{n_i-1}^i, y, x, x_2^v, ..., x_f^v = x_0$$

for some sequence s^v of length n_v, containing an element x_j^v for which $\left(x_j^v, x_{j+1}^v\right) = (x_0, y_0)$.

In this way, we construct a finite number of sequences, running over all the elements of all the former sequences. These new sequences obey all the conditions upon $(y_0, x_0) \in C(R)$, a contradiction. Conclude that $C(R_0) \cap P^{-1}(R_0) = \emptyset$.

2. $(y, x) = (x_0, y_0)$

   The proof of this case is similar to the previous one, and is left to the reader.

$\square$

Now we can state the result which characterizes the existence of a complete, transitive and convex extension.

Theorem 4 *A binary relation R has a complete, transitive and convex extension if and only if it is c-consistent, i.e. $C(R) \cap P^{-1}(R) = \emptyset$.*

Proof. From Lemma 6 and 7 together with Theorem 1 we know that a relation has a complete extension satisfying property c if and only if it is c-consistent. From Lemma 5 we know that this complete relation satisfies property c if and only if it is a complete transitive and convex relation. $\square$

If one desires to test wether a relation is c-consistent, the construction of the closure C may be to cumbersome, i.e. too many sequences have to be checked. We now develop a result that eliminates a large number of these sequences. We establish this in two steps. First we show that c-consistency is equivalent to the condition of c'-consistency (defined below) and next we show that c'-consistency is equivalent to the condition of c''- consistency (also defined below).

We define the closure $C'(R)$ as:

Definition 12 *$(x, y) \in C'(R)$ if $x = y$ or there is a finite sequence $s = x_1, ..., x_n$ of elements in X such that $x_1 = x$, $x_n = y$ and for all $i = 1, ..., n-1$ either*

$$(x_i, x_{i+1}) \in R$$

or

$$x_i \in V(A_i)$$

where A_i contains x_{i+1} and a collection of elements of the sequence s.

The definition of the C'-closure only uses one sequence, in stark contrast to the definition of C-consistency which uses a finite number of sequences. However, we have the following result.

Lemma 8 *A relation is c-consistent if and only if it is c′ consistent, i.e. $C(R) \cap P^{-1}(R) = \emptyset \leftrightarrow C'(R) \cap P^{-1}(R) = \emptyset$*

Proof. It is easy to see that c-consistency implies c'-consistency. To see the reverse, assume that R is not c-consistent, i.e. $(x, y) \in C(R) \cap P^{-1}(R)$. From $(x, y) \in C(R)$, we get that there exist a number $n \in \mathbb{N}$ of finite sequences $s^1, ..., s^n$ of elements in X where each sequence begins with x, ends with y and for each nonterminal element x_j^i of a sequence s^i and its immediate successor x_{j+1}^i either $\left(x_j^i, x_{j+1}^i\right) \in R$ or $x_j^i \in V\left(A_j^i\right)$, where A_j^i contains x_{j+1}^i and a finite number of elements of the sequences $s^1, ..., s^n$.
Consider the compound sequence

$$s = s^1, s^2, ..., s^n$$

As $(y, x) \in R$, this compound sequence satisfies the conditions upon $(x, y) \in C'(R)$. However, $(y, x) \in P(R)$, a contradiction with c'-consistency. □

We can further simplify the condition of c'-consistency by the concept of c''-consistency.
Let S_R be the subset of X defined as

$$S_R = \{x \in X \mid \text{there is an } y \in X \text{ for which } (x, y) \in R \text{ or } (y, x) \in R\}.$$

We define the closure $C''(R)$ as:

Definition 13 *$(x,y) \in C'(R)$ if $x = y$ or there is a sequence $s = x_1, ..., x_n$ of elements in S_R with $x_1 = x$, $x_n = y$ and for all $i = 1, ..., n-1$ either*

$$(x_i, x_{i+1}) \in R$$

or

$$x_i \in V(A_i)$$

where A contains x_{i+1} and a number of elements of the sequence s.

As the sequence in the definition of the closure C'' uses a considerable smaller domain for its elements compared to the sequence in the definition of the closure C', it is a lot easier to verify if $(x,y) \in C''(R)$, then to verify if $(x,y) \in C'(R)$. We have the following nice result.

Lemma 9 *A relation R is c'-consistent if and only if it is c''-consistent*

Proof. It is easy to see that c'-consistency implies c''-consistency. To see the reverse let R be c''-consistent and assume on the contrary that $(x,y) \in C'(R) \cap P^{-1}(R)$ (this implies that $x, y \in S_R$). From the definition, there is a sequence $s = x_1, ..., x_n$ of elements in X, where $x_1 = x$, $x_n = y$ and for each $i = 1, ..., n-1$ either $(x_i, x_{i+1}) \in R$ or $x_i \in V(A_i)$, where A_i contains x_{i+1} and a number of elements from the sequence s.

Let A be the set consisting of all the elements in the sequence s. Consider the set $A - S_R$. If this set is empty, then immediately $(x,y) \in C''(R)$ contradicting c''-consistency. Hence, $A - S_R$ is non-empty. For each element $x_i \in A - S_R$, we have that $x_i \in V(A_i)$. If $A - S_R$ has q elements, we have q equations, hence each element of $A - S_R$ can be expressed as a convex combination of elements in $A \cap S_R$. To see this, notice that each element $x \in A - S_R$ can be expressed as a convex combination of elements of $A - \{x\}$. In this way, the convex set spanned by the set A is equal to the convex set spanned by $A - \{x\}$. Consider a second element of $A - S_R$, e.g. x'. Let

$$x = \sum_i \alpha_i x_i + \alpha x', \sum_i \alpha_i + \alpha = 1, x_i \in A - \{x, x'\},$$
$$x' = \sum_i \beta_i x_i + \beta x, \sum_i \alpha_i + \alpha = 1, x_i \in A - \{x, x'\}.$$

Substitution gives us

$$x = \sum_i \frac{(\alpha_i + \alpha\beta_i)}{(1-\alpha\beta)} x_i.$$

And as $\sum\limits_i \frac{(\alpha_i+\alpha\beta_i)}{(1-\alpha\beta)} = \frac{\sum\limits_i \alpha_i + \sum\limits_i \alpha\beta_i}{1-\alpha\beta} = 1$ and $\frac{(\alpha_i+\alpha\beta_i)}{(1-\alpha\beta)} > 0$, we have that x can be expressed as a linear combination of elements in $A - \{x, x'\}$ and similarly for x'. Hence, the convex set spanned by the elements in A equals the convex set spanned by the elements in $A - \{x, x'\}$. Iteration over all elements in $A - S_R$, gives us that each element of $A - S_R$ is in the convex set spanned by the elements in $A \cap S_R$.

Consider following algorithm,

1. Set $j = 0$.
2. Set $y_1 = x_1$.
3. If $y_{j+1} = x_n$ stop, else increase j by one, i.e. $j := j + 1$.
4. For $y_j = x_i$, if $(x_i, x_{i+1}) \in R$, we set $y_{j+1} = x_{i+1}$ and we go back to step 3.
5. For $y_j = x_i$, if $x_i \in V(A_i)$ and $x_{i+1} \in S_R$, we set $y_{j+1} = x_{i+1}$. As every element in $A - S_R$ can be expressed as a linear combination of elements in $A \cap S_R$, by substitution $y_j \in V(A'_i)$, where $A'_i \subseteq A \cap S_R$ and contains y_{j+1}. Go back to step 3.
6. For $y_j = x_i$, if $x_i \in V(A_i)$ and $x_{i+1} \in A - S_R$, we have that $x_{i+1} \in V(A_{i+1})$. By substitution $x_i \in V(A'_i)$, where A'_i contains x_{i+2}. As $y \in S_R \cap A$, continuing this way, there must be a smallest v such that $x_i \in V(A^v_i)$, where A^v_i contains x_{i+v} and $x_{i+v} \in S_R \cap A$. We set $y_{j+1} = x_{i+v}$. As every element in $A - S_R$ can be expressed as a linear combination of elements in $A \cap S_R$, by substitution $y_j \in V\left(A^{v+1}_i\right)$, where A^{v+1}_i contains x_{i+v} and $A^{v+1}_i \subseteq A \cap S_R$. Go back to step 3.

As A is finite, this algorithm must stop in finite time. Every element in $A \cap S_R$ corresponds to an element y_j for a certain j. Notice that the finite sequence $s' = y_1, y_2, ...$ satisfies the requirements upon $(x, y) \in C''(R)$, a contradiction with c''-consistency of R. □

A straightforward result of Lemma 8 and 9 is that R has a complete, transitive and convex extension if and only if it is c''-consistent. This last condition is a lot easier to verify than the condition of c-consistency.

It would have saved us a lot of work if we would have started this section with the closures C' or C'' instead of the technical cumbersome definition of closure C. However, it turns out that the closures C' and C'' are not non-comparable extendible. As such, we are unable to use Theorem 1 directly on these closures. This indicates that, at least for the property of convexity, the requirement of non-comparable extendibility is stronger than necessary for Theorem 1 to be valid.

Let us return to the example in the introduction. We have that $x_2 = 0.5x_3 + 0.25x_4 + 0.25x_6$ and that $x_5 = 0.5x_6 + 0.25x_4 + 0.25x_3$. Consider the sequence $x_1, x_2, x_3, x_4, x_1, x_5, x_6, x_4$. We see that this sequence contradicts c''-consistency. Therefore, we must reject the hypothesis that there exist a convex ordering extension of the observed preferences relation.

3.3 Homothetic ordering extensions

Again, let our universal set X be a subset of $\mathbb{R}^m$. Further assume that X is homothetic closed, i.e. if $x \in X$, then for each $\alpha \in \mathbb{R}^+_0$, $\alpha x \in X$. Again this domain is stronger than necessary, but we retain the assumption because the increase in generality does not weigh up against the increase in notational complexity.

Definition 14 *A binary relation R on X is homothetic (property h) if*

$$(x,y)\in R \text{ (resp. } P(R)) \text{ and } \alpha\in\mathbb{R}_0^+ \text{ implies } (\alpha x,\alpha y)\in R \text{ (resp. } P(R)).$$

It turns out that homotheticity is a lot easier to analyze in accordance with monotonicity.

Definition 15 *A relation R is monotonic (property m) if*

$$x\geq y \text{ implies } (x,y)\in R.$$

In this section, we will join the properties of homotheticity, monotonicity and transitivity. As most economic relations are required to be monotonic, this is not a very stringent condition. Property h, m and t together are for notational simplicity written as property $\bar{h}$. We then write $\bar{h}(R)=1$ if and only if $h(R)=1$, $m(R)=1$ and $t(R)=1$. As all three properties are closed under intersection, the joint property is also closed under intersection.

The *homothetic, transitive and monotonic closure* of a relation R, $\bar{H}(R)$, is defined as:

Definition 16 $(x,y)\in\bar{H}(R)$ *if there is a sequence $s=x_1,x_2,...,x_n$ of elements in X such that $x=x_1$, $y=x_n$ and for all $i=1,2,...,n-1$ either*

$$x_i\geq x_{i+1},$$

or there is an $\alpha_i\in\mathbb{R}_0^+$, such that

$$(\alpha_i x_i,\alpha_i x_{i+1})\in R.$$

Notice that $\bar{H}(R)$ is reflexive. We begin by observing that the closure $\bar{H}$ indeed corresponds to the property $\bar{h}$.

Lemma 10

$$\bar{H}(R)=\left\{Q\supseteq R\,\middle|\,\bar{h}(Q)=1\right\}$$

Proof: straightforward. □

We are still left to show that $\bar{H}$ is algebraic and non-comparable extendible.

Lemma 11 $\bar{H}$ *is an algebraic, non-comparable closure operator.*

Proof. To see that $\bar{H}$ is algebraic, let $(x,y)\in\bar{H}(R)$. This means that either $x=y$ or there is a sequence $s=x_1,...,x_n$ of elements in X such that $x=x_1$,$x_n=y$ and for all $i=1,2,...,n-1$: either $x_i\geq x_{i+1}$ or there is an $\alpha_i\in\mathbb{R}_0^+$ such that $(\alpha_i x_i,\alpha_i x_{i+1})\in R$. For $x_i\geq x_{i+1}$, define $\alpha_i=1$ and let A collect all the elements $\alpha_i x_i$ and $\alpha_i x_{i+1}$. We have that $(x,y)\in H(R\cap A\times A)$ and $R\cap A\times A$ is a finite subset of R.

To see that $\bar{H}$ is non-comparable extendible, let $R = \bar{H}(R)$ and let $(x_0, y_0) \in N(R)$. Set $R_0 = R \cup \{(x_0, y_0)\}$ and assume on the contrary that $(x, y) \in \bar{H}(R_0) \cap P^{-1}(R_0)$.

This means that there is a finite sequence $s = x_1, ..., x_n$ of elements in X for which $x = x_1$, $y = x_n$ and for all $i = 1, 2, ..., n-1$, $x_i \geq x_{i+1}$ or $(\alpha_i x_i, \alpha_i x_{i+1}) \in R_0$ for some $\alpha_i \in \mathbb{R}_0^+$. If for all $i = 1, 2, ..., n-1$, where $(\alpha_i x_i, \alpha_i x_{i+1}) \in R_0$, also $(\alpha_i x_i, \alpha_i x_{i+1}) \in R$, then $(x, y) \in \bar{H}(R) = R$, in contradiction with $(y, x) \in P(R_0)$. Conclude that there is at least one x_i such that $(\alpha_i x_i, \alpha_i x_{i+1}) = (x_0, y_0)$.

Therefore, there is a finite set $S = \{\beta_1, ..., \beta_q\}$ of elements in $\mathbb{R}_0^+$ such that for all $i = 1, ..., q-1$, $\left(\frac{1}{\beta_i} y_0, \frac{1}{\beta_{i+1}} x_0\right) \in H(R)$, and $\left(\frac{1}{\beta_q} y_0, \frac{1}{\beta_1} x_0\right) \in \bar{H}(R)$. Take the smallest value from the set S, say β_j. For $j > 1$, by homotheticity of R: $\left(y_0, \frac{\beta_{j-1}}{\beta_j} x_0\right) \in R$ and by monotonicity $\left(\frac{\beta_{j-1}}{\beta_j} x_0, x_0\right) \in R$. By transitivity of R, we get that $(y_0, x_0) \in R$. This is in contradiction with $(x_0, y_0) \in N(R)$. If $j = 1$, we have that $\left(y_0, \frac{\beta_q}{\beta_1} x_0\right) \in R$ and $\left(\frac{\beta_q}{\beta_1} x_0, x_0\right) \in R$ and again by transitivity of R: $(y_0, x_0) \in R$. □

Our characterization result for complete, transitive, homothetic and monotonic extensions reads as follows.

Theorem 5 *A binary relation has a extension satisfying property $\bar{h}$ if and only if it is $\bar{h}$-consistent, i.e. $\bar{H}(R) \cap P^{-1}(R) = \emptyset$.*

Proof. Using Lemma 10 and 11 together with Theorem 1, we have that a relation has a complete extension satisfying property $\bar{h}$ if and only if it is $\bar{h}$-consistent. □

3.4 Complete and monotonic extensions

The last part of this section focusses on the properties of monotonicity and strict monotonicity. Again, we assume that X is a subset of $\mathbb{R}^n$.

We recall that a relation R on X is *monotonic* (property m) if

$$x \geq y \text{ implies } (x, y) \in R.$$

Definition 17 *A relation R on X is strict monotonic (property s) if in addition to monotonicity also*

$$x > y \text{ implies } (x, y) \in P(R).$$

Let us first focus on the property of monotonicity. Let

$$\bar{R} = R \cup \{(x, y) \,|x \geq y\}.$$

and for a closure K, let the operator $\bar{K}$ be defined as

$$\bar{K}(R) = K(\bar{R}).$$

Immediately, we have the following result on the operator $\bar{K}$.

Lemma 12 *If K is a non-comparable extendible, algebraic closure operator, then the operator $\bar{K}$ is also a non-comparable extendible, algebraic closure operator.*

Proof. To show that $\bar{K}$ is a closure operator, we show that it satisfies all four properties. Property 1 follows from $R \subseteq \bar{R} \subseteq K(\bar{R}) = \bar{K}(R)$. Property 2 follows from the fact that $R \subseteq Q$, implies $\bar{R} \subseteq \bar{Q}$. Property 3 is a consequence of $\bar{K}(\bar{K}(R)) = K(\bar{K}(R)) = K(K(\bar{R})) = K(\bar{R}) = \bar{K}(R)$ and property 4 follows from $\bar{K}(R) = K(\bar{R}) = K(S) = \bar{K}(V)$, with S a finite subset of $\bar{R}$ and $V = S - (\bar{R} - R)$.
We are left to show that $\bar{K}$ is non-comparable extendible. Let $R = \bar{K}(R)$ and $(x_0, y_0) \in N(R)$. We have that $\bar{R} \subseteq K(\bar{R}) = R$, hence R is monotonic and by definition $R_0 = \bar{R}_0$. We know that $R_0 = R \cup \{(x_0, y_0)\} \subseteq \bar{K}(R_0)$, so let $(x, y) \in P(R_0) = P(\bar{R}_0)$. The closure K is non-comparable extendible, hence $(x, y) \in P(K(R_0)) = P(K(\bar{R}_0)) = P(\bar{K}(R_0))$.
□

From this lemma we have that with every property k which is closed under intersection, we can associate a property $\bar{k}$ which is also closed under intersection and represents both properties k and m. We have the following result on the existence of a complete extension satisfying $\bar{k}$.

Theorem 6 *If K is a non-comparable extendible closure operator, then a binary relation has a complete extension satisfying properties k and m if and only if it is $\bar{k}$-consistent, i.e. $\bar{K}(R) \subseteq P^{-1}(R) = \emptyset$.*

Proof. We know from Lemma 12 that $\bar{K}$ is an algebraic, non-comparable extendible closure operator. Hence, from Theorem 1, $\bar{k}$-consistency is necessary and sufficient for the existence of a complete and $\bar{k}$-closed extension. It follows from the definition that this extension satisfies both properties k and m. To see the converse, assume on the contrary that R^* satisfies both k and m, and is a complete extension of R, and assume that $(x, y) \in \bar{K}(R) \cap P^{-1}(R)$. As $\bar{R} \subseteq \bar{R}^*$, we have that $\bar{K}(R) \subseteq \bar{K}(R^*) = R^*$. Therefore, we have that $(x, y) \in R^*$ and $(y, x) \in P(R^*)$, a contradiction. □

Let us now focus on the property s.

Theorem 7 *A binary relation has a complete extension that satisfies properties s and k if and only if,* $(x,y) \in \bar{K}(R)$ *implies* $(y,x) \notin P(R)$ *and* $y \not> x$*.*

Proof. Let R be $\bar{k}$-consistent. By Lemma 2, we have that $\bar{K}(R)$ is an extension of R. First we show that $\bar{K}(R)$ is strict monotonic. We have that $x > y$ implies $(x,y) \in \bar{R}$, and so $(x,y) \in \bar{K}(R)$. If also $(y,x) \in \bar{K}(R)$, we have a contradiction with the requirement that $x \not> y$. Conclude that $\bar{K}(R)$ is strict monotonic. The relation $\bar{K}(R)$ satisfies property $\bar{k}$, hence it is $\bar{k}$-consistent. From Theorem 6 it has a monotonic, complete extension satisfying property k, lets say R^*. If $x > y$, we have $(x,y) \in P\left(\bar{K}(R)\right)$ giving $(x,y) \in P(R^*)$. Conclude that R^* is strict monotonic.

To see the converse, let R^* be a strict monotonic, complete extension of R satisfying property k . From this, R is $\bar{k}$-consistent. Now assume on the contrary that $(x,y) \in \bar{K}(R)$ and $y > x$. Then we have $(x,y) \in K(R^*) = R^*$, contradicting strict monotonicity of R^*. □

The last two theorems can be used in combination with the previous results. For example; a relation has a strict monotonic and convex ordering extension if and only if $(x,y) \in \bar{C}(R)$ implies $(y,x) \notin P(R)$ and $y \not> x$. Another example is: a relation has a strict monotonic and homothetic ordering extension if and only if $(x,y) \in \bar{H}(R)$ implies $(y,x) \notin P(R)$ and $y \not> x$.

4 Closure rationalizability

This section, applies the results to the rationalizability problem.

Let X be a universal set of alternatives and let Σ be a set of nonempty subsets of X. A choice function F is a correspondence

$$F : \Sigma \to X : S \to F(S) \subseteq S,$$

such that for all $S \in \Sigma$, $F(S)$ is nonempty. Let k be a property, closed under intersection. We assume that property k includes transitivity, i.e. for all $R \in \mathcal{R}$, if $t(R) = 1$, then $k(R) = 1$.

Definition 18 *A choice function is said to be k-rationalizable if there exist a complete binary relation R that satisfies property k, so that for all* $S \in \Sigma$*,*

$$F(S) = \{x \in S \,|(x,y) \in R \text{ for all } y \in S\},$$

i.e. the elements chosen from S, are top ranked according to R.

Definition 19 *The revealed preference relation* R_v *is given by* $(x,y) \in R_v$ *if there is a set* $S \in \Sigma$*, so that* $x \in F(S)$ *and* $y \in S$*. If also* $y \notin F(S)$ *we say that x is strictly revealed preferred to y and write* $(x,y) \in P_v$*.*

The next result gives a characterization for k-rationalizability.

Theorem 8 *If the closure operator K is non-comparable extendible, then a choice function is k-rationalizable if and only if $K(R_v) \cap P_v^{-1} = \emptyset$.*

Proof. Let F be k-rationalizable, with a rationalization R^*. It is easy to see that $R_v \subseteq R^*$ and that $P_v \subseteq P(R^*)$ (notice that R^* is transitive). If $(x, y) \in K(R^*)$, we must have that $(y, x) \notin P_v$. Indeed, otherwise we would have that $(x, y) \in R^*$ and $(y, x) \in P(R^*)$, a contradiction.

To see the reverse, let $K(R_v) \cap P_v^{-1} = \emptyset$. It is easy to see that $P_v = P(R_v)$, hence R_v is k-consistent. By Theorem 1, R_v has a complete extension that satisfies property k, lets say R^*. We show that R^* rationalizes F. If $x \in F(S)$, then by definition $(x, y) \in R_v$ for all $y \in S$, hence $(x, y) \in R^*$ for all $y \in S$. If $x \notin F(S)$ then either $x \notin S$, or there is an $y \in S$ so that $(y, x) \in P_v$. In either case $x \notin \{z \in S \,|\, (z, y) \in R^* \text{ for all } y \in S\}$. □

Theorem 8 is immediately applicable to the properties t, c and $\bar{h}$ (notice that all properties imply transitivity). For this, let the universal set X be a convex and homothetic subset of $\mathbb{R}^n$. We have the following result, which we give without proof.

Theorem 9 *A choice function F is rationalizable by:*

- *an complete and transitive relation if and only if $(x, y) \in T(R_v)$ implies $(y, x) \notin P_v$,*
- *a complete, transitive, monotonic and homothetic relation if and only if $(x, y) \in H(R_v)$ implies $(y, x) \notin P_v$,*
- *a complete, transitive and convex relation if and only if $(x, y) \in C(R_v)$ implies $(y, x) \notin P_v$,*
- *a complete, transitive, convex and monotonic relation if and only if $(x, y) \in \bar{C}(R_v)$ implies $(y, x) \notin P_v$,*
- *a complete, transitive,convex and strict monotonic relation if and only if $(x, y) \in \bar{C}(R_v)$ implies $(y, x) \notin P_v$ and $y \not> x$,*
- *a complete, transitive, homothetic an strict monotonic relation if and only if $(x, y) \in H(R_v)$ implies $(y, x) \notin P_v$ and $y \not> x$.*

Naturally, we can replace c-consistency with c' or c''-consistency
If a choice function is determined by real data, Theorem 9 provides an exact test for the following hypothesis.

H_0 : the individual has a complete, transitive, (strict monotonic, monotonic) and (convex, homothetic) preference relation.

The test is exact in the sense that the probability of rejecting H_0, when it is true, is zero. This in contrast to other, statistical, test procedures.

5 Conclusion

In this paper, we discussed the existence of closed and complete extensions. Our main result, Theorem 1, states that, given an algebraic, non-comparable extendible closure operator K, a binary relation has a complete relation satisfying property k if and only if it is k-consistent.

We used this theorem to characterize the existence of complete, transitive, convex, (strict) monotonic and homothetic extensions. Our result is an extension to the literature since we only impose domain restrictions insofar they are necessary for the properties to be well defined. We used our results to characterize the choice functions which are rationalizable by a complete relation satisfying properties which can be represented by an algebraic, non-comparable extendible closure operator.

Theorem 1 provides a general framework for the existence of complete binary extensions which satisfies additional properties which are closed under intersection. The properties discussed in this research are not the only ones that fit this framework (e.g. separability). On the other hand, it should be mentioned that not all interesting economic properties can be fitted into our framework. As an example we briefly present two such properties.

Consider a binary relation R. This relation is defined to be of dimension n if there exist n complete and transitive relations, $R_1, ..., R_n$ such that $R = \bigcap_{i=1}^{n} R_i$. A binary relation of dimension n can be seen as the Pareto relation from a group of n people. Recently, there has been some research on the testable implications of this property in case $n = 2$ (see Bossert and Sprumont (2002), Sprumont (2001)). The property of being of dimension n is not closed under intersection, and therefore can not be dealt with in our framework.

A binary relation R is continuous if the sets $L(y) = \{x \in X \,|(y, x) \in R\}$ and $U(y) = \{x \in X \,|(x, y) \in R\}$ are closed for all $y \in X$. This property is closed under intersection (as the intersection of any set of closed sets is also closed) but it is not algebraic. Therefore, the property of continuity cannot be dealt with in our framework[3].

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[3]For a characterization of continuous ordering extensions, see Herden and Pallack (2002)

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