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WORKING PAPER

Don't Fall from the Saddle: the Importance of Higher Moments of Credit Loss Distributions

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Abstract

The original Panjer recursion of the CreditRisk+ model is said to be unstable and therefore to yield inaccurate results of the tail distribution of credit portfolios. A much-hailed solution for the flaws of the Panjer recursion is the saddlepoint approximation method. In this paper we show that the saddlepoint approximation is an accurate and robust tool only for relatively homogenous credit portfolios with low skewness and kurtosis of the loss distribution. However, often credit portfolios are heterogeneous with large skewness and kurtosis. We show that for such portfolios the commonly applied saddlepoint approximations (the Lugannani-Rice and the Barndorff-Nielsen formulas) are not reliable. Moreover, when applied to such credit portfolios, the Lugannani-Rice formula is fragile. We explain it by the dependence of the high-order standardized cumulants and the relative error on the saddlepoints. The more the cumulants and the relative error vary, the less accurate the saddlepoint approximation is. Hence, the saddlepoint approximation is not a universal substitute to the Panjer recursion algorithm.

JEL.
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1. Introduction

The CreditRisk+ (CR+) model of Credit Suisse First Boston (CSFB) has several advantages over its rival credit risk models such as KMV and CreditMetrics. The CR+ model offers a full analytic description of the portfolio loss of a credit portfolio, allowing a quick calculation of the loss distribution thereby avoiding time-consuming Monte Carlo simulations. The model also requires a relatively limited dataset (Kurth, et al., 2002). However, Wilde (2000), Gordy (2002), and Haaf, Reiß and Schoenmakers (2003) draw attention to the numerical fragility and therefore inaccuracy of the Panjer recursion algorithm of CR+. Its numerical instability arises from an accumulation of numerical round-off errors due to the summation of numbers of similar magnitude but opposite sign. The larger the number of independent risk factors, the larger the number of obligors and the smaller the standardized loss unit, the longer the polynomials in the recurrence equation and, hence, the larger the possibility for round-off errors to accumulate. As an alternative to the Panjer recursion algorithm, Gordy (2002) introduces a saddlepoint approximation (SPA) for fast and accurate computation of tail percentiles of the loss distribution in CR+. He finds that the SPA is extremely fast, accurate and robust for large portfolios with complex risk factor structures, exactly in the situations for which the Panjer recursion algorithm fails. Nevertheless, the SPA is less accurate in situations where the Panjer recursion is fast and reliable such as small portfolios with only one risk factor. Feuerverger and Wong (2000) test the accuracy of the Lugannani-Rice SPA introduced by Gordy (2002) as well as the alternative Barndorff-Nielsen SPA (Jensen, 1995) for large complex portfolios whose payoff functions contain linear (direct holdings in the underlying risk factor assets) and nonlinear (due to derivative securities) terms. They conclude that both SPAs are fast and accurate for any portfolio for which the risk factors are normally distributed with correctly specified covariance and for which a delta-gamma approximation of the nonlinear terms is appropriate.

Although SPA is said to be extremely accurate in the tails and even exact for normal, gamma and inverse Gaussian distributions (Daniels, 1980), some notes of caution are expressed in the literature on their reliability in other cases. Booth and Wood (1995), Beran and Ocker (2003), Studer (2001), and Maryno and Nyfeler (2003) warn that the standard normal SPA should be applied cautiously as it produces large errors for some distributions.

Our purpose is to provide users of CR+ with a set of diagnostics to identify beforehand when the SPA is prone to failure. We test the accuracy and robustness of the Lugannani-Rice (LR) and the Barndorff-Nielsen (BN) formulas on a set of real-life credit portfolios. In contrast to Gordy (2002), we find that there exists a class of credit portfolios for which both SPAs perform poorly, even in a simple one risk factor setting. Both formulas yield accurate approximations of the credit loss distributions for homogenous portfolios only, where homogenous portfolios are understood to be portfolios in which the maximum exposure to a single obligor is less than 1% of total book value. Their credit loss distributions exhibit low skewness and kurtosis. However, in practice credit portfolios are often heterogeneous and characterized by large skewness and kurtosis. We show that applying the LR and BN formulas to such credit portfolios yields totally unreliable approximations in the tails of the loss distribution. Moreover, the LR formula fails in the sense that sometimes negative probabilities are generated. Large relative errors of SPA are explained by the dependency on the saddlepoint, θ, of the third $\zeta_3(\theta)$ and the fourth $\zeta_4(\theta)$ standardized cumulants and the relative error (see section 3). We find that the larger $\zeta_3(\theta)$, $\zeta_4(\theta)$ and hence the relative error are at $\theta = 0$, the less accurate the SPA is. Therefore, we warn against applying SPA on credit portfolios without first checking the third and the fourth moments of the loss distribution. At the same time we corroborate Gordy's (2002) results that under a simple one risk factor specification of the CR+ the Panjer recursion algorithm gives accurate results for both homo- and heterogeneous portfolios, regardless of the third and the fourth moments of the credit loss distributions.

The plan of the paper is as follows. The next section gives a short explanation of SPA. In the third section we present an overview of the literature that documents the inaccuracy and even failure of SPA in some cases. In the fourth section we test the accuracy of the Panjer recursion algorithm, the LR and the BN formulas on five real-life credit portfolios with loss distributions exhibiting varying degrees of skewness and kurtosis. The fifth section concludes.

2. Saddlepoint approximations

SPAs, or saddlepoint expansions, were derived in the 1930s and were fully developed in the 1980s. The application of SPAs historically began with risk theory for calculating risk premiums in insurance. Nowadays SPAs are applied to credit risk modelling (Feuerverger and Wong (2000), Martin et al. (2001), Gordy (2002 and 2004) and Glasserman (2003)), catastrophe insurance and operational risk management (Schimiddi and Barndorff-Nielsen (1995)).

The intuition behind the SPA is as follows. The Edgeworth expansion gives an approximation of the density of a centered random variable for which the closed-form solution is not known. In general, Edgeworth expansions are found to work well at the center of a distribution but not at the tails of a distribution. The trick of the SPA is to use the Edgeworth expansion precisely where it works well, in the center of a distribution. The density of the original distribution $f(x)$ is estimated by 'tilting' $f(x)$ to a new distribution $f(\theta;x)$ which is centered around x. This new tilted distribution is then approximated by the Edgeworth expansion and finally the mapping is inverted to obtain the approximation of $f(x)$.[2]

Let f_n be the density of $\bar{X} = \left(X_1 + ... + X_n\right)/n$, where the X_i are i.i.d. random variables. The low-order SPA of f_n has the following form

$$f_n(x) = \exp\left(n\left(K\left(\theta\right) - \theta x\right)\right) \cdot \sqrt{n/2\pi\sigma^2\left(\theta\right) \cdot}\left(1 + O\left(n^{-1}\right)\right), \tag{1}$$

where n denotes the number of random variables, $\sigma^2\left(\theta\right)$ the variance of the tilted distribution, $K\left(\theta\right)$ the cumulant generating function (cgf) and O the relative error term. The saddlepoint θ is chosen in such a way that the first cumulant (i.e. the mean) equals x – the point for which we want to know the density. The standard high-order SPA formula is as follows:

$$f_n(x) = \exp\left(n\left(K\left(\theta\right) - \theta x\right)\right) \cdot \sqrt{n/2\pi\sigma^2\left(\theta\right) \cdot}\left(1 + \frac{1}{n}\left(\frac{\zeta_4(\theta)}{8} - \frac{5\zeta_3(\theta)^2}{24}\right) + O\left(n^{-2}\right)\right), \tag{2}$$

where ζ_i are the ith standardized cumulants defined as $\zeta_i(\theta) = \kappa_i(\theta)/\kappa_2^{i/2}(\theta)$, with κ_i the ith cumulant. Tail probabilities are given by the following formula

[2] In this paper only final formulas for SPA which are useful for the further discussion are presented. An extensive discussion of SPA can be found in the textbook by Jensen (1995).

$$\Pr\left(\bar{X} \geq x\right) = \frac{\exp\left(n\left(K\left(\theta\right) - \theta x\right)\right)}{\sqrt{n}\theta\sigma\left(\theta\right)} \times$$
$$\left(B_0(\lambda) + \frac{\zeta_3(\theta)}{6\sqrt{n}} B_3(\lambda) + \frac{1}{n}\left(\frac{\zeta_4(\theta)}{8} B_4(\lambda) - \frac{\zeta_3(\theta)^2}{72} B_6(\lambda)\right) + O\left(B_0(\lambda) n^{-3/2}\right)\right), \tag{3}$$

where $B_i(\lambda)$ are the Esscher functions (see Jensen, 1995).

The SPA (3) for tail probabilities is cumbersome and does not have a simple relation to other well-known quantiles in statistics. Practitioners often need simple formulas expressed in interpretable quantiles, such as the LR approximation (Jensen, 1995):

$$\Pr\left(\bar{X} \geq x\right) = 1 - G(x) \approx 1 - \Phi(\omega) + \phi(\omega)\left(\frac{1}{\upsilon} - \frac{1}{\omega}\right), \tag{4}$$

where $G(x)$ is the cumulative density function (cdf) of random variable $\bar{X}$, Φ and ϕ denote resp. the cdf and density of the standard normal distribution, $\omega = \operatorname{sign}\left(\theta\right)\sqrt{2n\left[\theta x - K\left(\theta\right)\right]}$ and $\upsilon = \theta\sqrt{nK''(\theta)}$. In the last expression $K''(\theta)$ is the second derivative of K w.r.t. θ.

An alternative to the LR formula is the BN formula (Jensen, 1995):

$$\Pr\left(\bar{X} \geq x\right) = 1 - G(x) \approx 1 - \Phi\left(\omega + \frac{1}{\omega}\log\frac{\upsilon}{\omega}\right) . \tag{5}$$

In order to avoid root-solving for θ, a strategy based on interpolation is applied (Gordy, 2002). This strategy is efficient when it is required to calculate VaRs for several probabilities. At first, the upper bound $\hat{\theta}$ is computed[3] and then a fine grid of 1,000 values in the open interval (0, $\hat{\theta}$) is formed. At each point in the grid we calculate the pairs of losses and corresponding probabilities and then interpolate to find the loss corresponding to a concrete target solvency probability.

SPAs are of interest in many different applications as they give good approximations to tail probabilities. However, they cannot be used to approximate any distribution. First, the cgf of the

[3] The procedure of computing $\hat{\theta}$ for CR+ is described in Gordy (2002).

distribution must have a tractable form (Jensen (1995), Studer (2001), Gordy (2002)). Second, for many distributions its accuracy can be questioned. This issue is discussed in the next section.

3. Cases where saddlepoint approximations fail

Wood, et al. (1993), and Booth and Wood (1995) study the reliability of the standard LR and BN formulas. Booth and Wood apply both approximations to a first passage time of a random walk with drift to a fixed boundary, which is characterized by the inverse Gaussian distribution, and find that for specific parameter values both approximation formulas give extremely inaccurate tail probability estimates. Moreover, they find that the LR approximation sometimes yields negative densities.

Studer (2001) examines the performance of the standard low-order SPA (1) on an example of the Normal Inverse Gaussian (NIG) Lévy process and also finds that it is not able to accurately approximate the distribution. The higher order saddlepoint approximation (2) works better, but the relative errors are still large over an interval capturing 99.8% of the mass of the distribution.

Another example of the standard SPA's inaccuracy is given by Beran and Ocker (2003). They consider credit portfolios with a few exceptionally high potential loss values and thus exhibiting bimodality in credit loss distribution. They find that, although the standard SPA captures the overall shape of the loss distribution of such portfolios, it totally ignores the fat and humped shape of the tail and, hence, significantly underestimates the credit risk.

These cases all have in common that the distributions for which the SPA yields inaccurate results have large third and fourth standardized cumulants. In fact, Studer (2001) concludes that the standard SPA can give very good results only for well-behaved distributions, i.e. distributions whose fourth cumulant is not too large. For distributions with large higher cumulants, the Edgeworth expansion is found to yield imprecise probability estimates in the center of its distribution (exactly where it should work well) (Jensen, 1995). Therefore, the large higher standardized cumulants result in a large relative error for the probability estimate of the SPA. Moreover, a less documented but elementary condition for the SPA to give accurate estimates is for the relative error to be independent of the saddlepoint (θ). The rest of this section elaborates on this condition and analyses whether it is fulfilled for the SPA of the CR+ specification.

Thus, a crucial condition for the SPA to give accurate results is for the relative error (A(θ) = $\frac{1}{n}\left(\frac{\zeta_4(\theta)}{8}-\frac{5\zeta_3(\theta)^2}{24}\right)$+…) to be independent of the saddlepoint approximation θ (or A(θ) = A) (Barndorff-Nielsen and Cox (1979), Daniels (1954, 1980), and Jensen (1995)). Daniels (1980) proves that for only three distributions – the normal, the gamma and the inverse-normal distribution – the SPA is exact, i.e. SPA estimates give for all n exactly the densities of the distributions. For the normal distribution the higher standardized cumulants (and thus the relative error) equal zero, hence the relative error and the standardized cumulants are independent of the saddlepoint. The relative error for the gamma distribution remains constant (but not zero) for all possible values of θ.[4] A special case is the inverse Gaussian distribution, Jensen (1995) shows that although the standardized cumulants $\zeta_j(\theta)$ depend on the saddlepoint θ and can even increase without bound, the SPA is exact, because the relative error is identical to zero for varying θ.

The condition of independence of the relative error on the saddlepoint does not hold for the one risk factor CR+ specification.[5] Limiting us to the leading term of the asymptotic expansion for the relative error

$$\left(\zeta_4(\theta)/8-5\zeta_3(\theta)^2/24\right), \tag{6}$$

we show in the appendix that the error term (6) depends on the saddlepoint θ.[6] Moreover, the error term (6) has the limit $-\sigma^2/12$, where σ^2 is the variance of the risk factor and $\hat{\theta}$ is the upper bound of the valid range of saddlepoints, calculated according to the inequality derived by Gordy (2002). In the appendix we also show that for the CR+ specification the higher order standardized cumulants $\zeta_3(\theta)$, $\zeta_4(\theta)$ depend on θ and have the following limits:

[4] Strictly speaking, the fact that the relative error is not zero implies that the *renormalized* SPA, i.e. the SPA multiplied by a constant such that it integrates to one, is exact.

[5] The conclusions derived here are also valid for the multifactor CR+, though the limits for the multifactor model differ from those derived for the one factor model. In this paper we do not derive the limits of the standardized cumulants and correction terms for the multifactor CR+ as the proof is messy and without additional insight.

[6] As the relative error is an asymptotic expansion specification (6) is not sufficient to prove the independence as the remaining part of the asymptotic expansion may eliminate any dependency. However, we restrict ourselves to the specification in (6) as in empirical applications this term will dominate.

$$\lim_{\theta\to\hat{\theta}} \zeta_3(\theta) = 2\sigma \quad \lim_{\theta\to\hat{\theta}} \zeta_4(\theta) = 6\sigma^2 . \tag{7}$$

In practice we are interested in distributions for which the SPA will not give exact results. Accuracy is the best thing to exactness. Daniels (1954) shows that several density specifications obey the already mentioned independence condition and for which the SPA yields accurate estimates (as those specifications approximate the normal or the gamma distributions for the saddlepoint $\theta \to \hat{\theta}$). More recent results from Daniels (1980) and Jensen (1995) suggest that investigating the behaviour of the third and the fourth cumulants and the leading correction term in the asymptotic expansion (6) across the valid range of θ may provide a useful diagnostic tool to ascertain the accuracy of the SPA.

In the next section we show empirically that the SPAs (LR and BN) for CR+ yield highly accurate estimates for these credit loss distributions of which the variation of the third and fourth standardized cumulants ($\zeta_3(\theta)$, $\zeta_4(\theta)$) remains limited across the valid range of θ. In these cases the standardized cumulants of the loss distributions are seen to be relatively small and to converge relatively fast towards their limits (7). However, in the cases where the standardized cumulants and the error term (6) vary strongly with the saddlepoint, the slower they converge to their limits and therefore the much less accurate the SPA is found to be. The credit loss distributions in these cases are characterized by large skewness and kurtosis.

The rule of thumb is as follows. First, compute the third and the fourth standardized cumulants at $\theta = 0$, i.e. for the original credit loss distribution. If the standardized cumulants and the correction term diverge significantly from their limits, then the SPA is likely to produce large relative errors and therefore should be substituted by another method (as for example the Panjer recursion).

4. The importance of higher moments for saddlepoint approximations

In this section we illustrate the importance of the third and the fourth moments on the accuracy of the VaR-estimates of the SPA for the CR+ model. The accuracy and robustness of the Panjer recursion and the SPA (both the LR and the BN formulas) are tested on a set of portfolios randomly sampled from a real-life credit portfolio. The large real-life credit portfolio consists of 3,000 exposures for which we know the exposure size and the unconditional probability of default, estimated by the bank's experts. Each of the five sampled credit portfolios consists of 1,000 obligors.

To introduce heterogeneity in the portfolios we increase the size of maximum exposure to a single borrower from 0.94% of total book value in portfolio 1 to 4.35% and 4.29% in portfolios 4 and 5 respectively. At the same time we gradually increase the difference between the first and the second largest exposures. For example, in the extreme cases of portfolios 4 and 5 the ratio between these two exposures is larger than 2, while in portfolio 1 it is about 0.5. Moreover, we keep the number of exceptionally high exposures restricted to no more than four. By doing so, we are able to increase dramatically the skewness and the kurtosis of the exposure size and the loss distributions. Summary statistics of exposure size and credit loss distributions are reported in table 1.

Insert table 1 around here.

The difference between the five portfolios is situated in the skewness and kurtosis of the loss distribution with minimal values for portfolio 1 and maximal values for portfolio 5. All portfolios, except portfolio 1, have almost identical values of expected loss and standard deviation in relative terms, but differ remarkably in their skewness and kurtosis. Portfolios 3 and 4 are built in such a way that they differ remarkably only in kurtosis. Checking only the skewness and the kurtosis of the exposure size distribution gives an incomplete picture of the riskiness of the portfolio. For example, although the skewness and the kurtosis of the exposure size distribution of portfolio 4 is larger than that of portfolio 5, the skewness and the kurtosis of the loss distribution show that portfolio 5 is likely to suffer larger losses. The histograms of the loss distributions are shown in figure 1. Due to the increase in heterogeneity, portfolios 4 and 5 have loss distributions with spikes far in the tails, marked by the circles.[7]

Insert figure 1 around here

[7] The distributions are estimated using a Monte Carlo simulation, cfr. infra.

Different levels of VaR are estimated on the one hand by the original Panjer recursion algorithm and on the other hand by the LR and the BN formulas. We restrict our analysis to a simple specification of the CR+ model, assuming exposure to only one systematic risk factor[8] with variance $\sigma^2 = 1$ and loss-given-default (LGD) equal to 0.3. Gordy (2002) shows that the larger the number of sectors, the larger the portfolio and the smaller the standardized loss unit, the longer the polynomials in the recurrence equation, so the greater the opportunity for round-off errors to accumulate and for the Panjer recursion algorithm to fail. However, for small and medium-sized portfolios (less than 5,000 obligors) and a simple one factor CR+ specification, the original Panjer recursion algorithm is stable and reliable. We specifically test the reliability of the Panjer algorithm when applied to portfolios with large third and fourth cumulants.

We will use as benchmark the loss distribution estimated using Monte Carlo simulation of the CR+ model with one systematic risk factor. This procedure has the advantage that we can also easily compute a 95% confidence interval around the VaR numbers. In the one factor specification the probability of default for obligor i conditional on the risk factor x is given by

$$p_i(x) = \overline{p}_i x , \tag{9}$$

where $\overline{p}_i$ is the unconditional default probability of obligor i (for example, given by rating agencies or bank experts).[9] The risk factor x is assumed to be an independent gamma variable with mean one and variance one. CR+ assumes that defaults follow a Poisson distribution with intensities $p_i(x)$. We therefore perform the Monte Carlo simulations in four steps:

1. Simulate the realization of the risk factor x from the gamma distribution $\Gamma(1, 1)$;
2. Compute the probability of default for each obligor from equation (9);
3. Simulate default events for each obligor from the Poisson distribution and calculate total portfolio loss in the portfolio;
4. Repeat this procedure N=100,000 times.

[8] The idiosyncratic risk is not included in the model. It is the simplest and at the same time the most prudent specification of CR+, because it assumes the highest correlation between the default probabilities. Inclusion of the idiosyncratic risk does not change the conclusions, but instead raises questions about the weights attached to the risk factors.

[9] The generalized K-factor specification of CR+ is given in Gordy (2002).

The error of a VaR estimate is calculated as a difference between the estimate and Monte Carlo estimate. Without confidence intervals the picture of the VaR error is incomplete. We derive explicit confidence intervals as described in Pritsker (1997). To construct a 95% confidence interval for the p^{th} percentile of a distribution generated by the Monte Carlo simulation, we solve for L and H such that $\Pr\left[V_L \le VaR \le V_H\right] = p$, where V_i, for i = 1,…, N, are the portfolio values generated by the simulation and sorted in ascending order. V_L and V_H are the bounds for the confidence interval. These bounds are chosen such that:

$$\begin{aligned} &\sum_{i=L}^{H-1} \binom{N}{i} p^i (1-p)^{N-i} \ge 0.95 \\ &\sum_{i=L+1}^{H-1} \binom{N}{i} p^i (1-p)^{N-i} \le 0.95. \end{aligned} \tag{10}$$

Moreover, H and L are chosen such that the confidence interval is as close to symmetric as possible.

In table 2 we report Monte Carlo, Panjer, LR and BN VaRs for all five portfolios and the probability that Monte Carlo loss will exceed these estimates. We show the results only for VaR_α at 90%, 95%, 97.5%, 99% and 99.5% levels, which are the most relevant quantiles in practice. To ease the comparison of the portfolios, we express VaR as a percentage of total book value. In portfolio 2 Monte Carlo losses exceed $\text{VaR}_{90\%}$ calculated by the Panjer recursion algorithm in 10.36% times. Ideally it should be 10%. We also test whether deviations from the Monte Carlo simulations are statistically significant.

We conclude that under the simple one factor specification the Panjer recursion algorithm indeed performs better than the LR and the BN formulas although its accuracy decreases for higher skewness and kurtosis of the loss distribution. This finding is consistent with Gordy (2002) who finds that the potential for errors (of the Panjer recursion) increases as the skewness of the loss exposure distribution is increased. Indeed, the Panjer recursion algorithm gives correct approximations in the tail of Portfolio 1, but in case of Portfolios 4 and 5 the approximations are correct only starting from α = 97.5%. As concerns the LR and the BN formulas, their accuracy deteriorates significantly when the skewness and the kurtosis of the loss distributions increase. For example, in portfolio 1, with minimum skewness and kurtosis, the deviations of VaR calculated by means of LR and BN formulas from Monte Carlo VaR are statistically insignificant. Starting from portfolio 3 both SPAs produce wrong estimates with statistically significant deviations over the whole tail of the loss distribution. There is no systematic pattern in the behavior of the LR and the BN formulas. In some cases the VaRs are significantly overestimated (portfolios 2 and 3), in others significantly underestimated (portfolios 4 and 5 for α = 90%). Hence, the users of the SPA are not always on the safe side, especially for heterogeneous portfolios highly inaccurate results are obtained.

Insert Table 2 around here

As a robustness check we test how close, or how far, the estimated distributions are from the distribution calculated via the Monte Carlo simulation. We estimate the goodness-of-fit by the Kolmogorov-Smirnov test. The results are summarized in the last column of table 2. For each portfolio we compare the tail distribution produced by the Panjer algorithm and the SPA against the Monte Carlo simulation. In not a single case can we reject the null hypothesis about the equality of distributions produced by the Panjer algorithm and Monte Carlo simulations. On the contrary, we reject the null hypothesis of equality of the distributions produced by the LR formula and Monte Carlo simulation for portfolios 4 and 5.

Our results are presented as Q-Q plots in figure 2 where the probabilities of exceeding Panjer, LR and BN VaRs are plotted against the probabilities according to Monte Carlo simulations for all five portfolios. The values of both axes decrease as we move from the origin, which indicates that we move further into the tail of the distribution. A good fit is indicated by values plotted on the diagonal (the full straight line), which means that the estimated probability of having a loss higher than VaR and the 'true' probability calculated via Monte Carlo simulations are equal. If the values plot below the diagonal, then the Monte Carlo probability exceeds the estimated and VaR is underestimated; if it is above, then it is overestimated. We see that probabilities calculated by the Panjer recursion algorithm are slightly underestimated but do not deviate much from the Monte Carlo probabilities. In fact, the accuracy increases as we move further into the tails. The probabilities calculated via the LR and the BN formulas deviate strongly from the diagonal. Moreover, in the case of LR the dots are scattered almost randomly in the Q-Q plot. This indicates that the LR approximation is fragile and cannot approximate the tails of the loss distributions of portfolios 4 and 5[10]. In fact, both SPA formulas are able to approximate correctly only the loss distribution of portfolio 1.

Insert Figure 2 around here

As we already mentioned in section 3, the accuracy of the SPA depends on how strongly the third and the fourth standardized cumulants vary when θ changes. In Figure 3 we plot the values of the cumulants and the relative error versus θ for the one systematic risk factor CR+ model with variance one. Under such assumptions the limits of $\zeta_3(\theta)$ and $\zeta_4(\theta)$ are 2 and 6 respectively.[11] The limit of the relative error is -0.083.[12] At $\theta = 0$ the third standardized cumulant of portfolio 1 is 2.026, the fourth is 6.117 and the relative error is -0.09.[13] They are initially very close to the limits, and converge fast as $\theta \to \hat{\theta}$. In fact, they are almost stable, therefore it is not surprising that the LR and the BN formulas work well in this case. In contrast to portfolio 1, the third and the fourth standardized cumulants of portfolios 4 and 5 are initially far from the limits[14] and vary a lot with θ, causing huge variation in the relative error. Therefore they converge to the limits very slowly causing large errors in the SPA.

Insert Figure 3 around here

In order to check whether these conclusions hold in a multifactor model, we increased the number of risk factors to 2 and 5. We came to exactly the same conclusions as in the one factor model.[15]

[10] The fragility of the LR approximation has also been reported by Booth and Wood (1995).
[11] According to eq. (7).
[12] According to eq. (6).
[13] Before calculating the standardized cumulants all exposures are grouped in 100 exposure bands. Therefore the standardized cumulants at $\theta = 0$ differ from the skewness and kurtosis reported in table 1, where the results are obtained without preliminary dividing portfolios into exposure bands.
[14] The third standardized cumulants of portfolios 4 and 5 are 3.9 and 4.9 respectively. The fourth standardized cumulants are 38.2 and 59.4 respectively.
[15] In order to save space, we do not report the figures of $\zeta_3(\theta)$ and $\zeta_4(\theta)$ and the relative error for multifactor CR+. These figures are available on request.

The instability of the LR formula can be completely understood when looking at the loss exceedance curve that shows for each loss value the probability of exceeding this loss. For a loss distribution to be a reliable approximation of the true loss distribution at least two properties of the loss exceedance curve have to hold. The first property is that the loss exceedance curve must be a non-increasing function of the loss value. It may be flat on some intervals implying that probability of loss occurring in those intervals is zero, but an increasing function would imply negative probabilities for some losses. If the first property does not hold then the interpolation to find the loss value x corresponding to $1\text{-}G(x)$ is impossible and the LR formula fails. The second property is of course that the loss exceedance curve can never have negative values, i.e. loss probabilities cannot be negative.

Insert Figure 4 around here

Figure 4 shows the loss exceedance curves of the first and the fifth credit portfolios produced by the LR formula.[16] The loss exceedance curve of the first portfolio obeys the above mentioned properties. However as for portfolio 5, the loss distribution given by the LR formula can never be a close approximation of the true loss distribution. The loss exceedance curve of the fifth portfolio is not continuously decreasing signaling that the credit loss distribution given by the LR formula is totally unreliable.

Several solutions have been brought up to increase the accuracy of the SPA. First, Studer (2001) points at higher-order SPA (2) as a partial solution for the large relative error given by the standard low-order SPA. Although this solution leads to a decrease in the relative error it does not solve the inaccuracy problem as often this error remains quite large. Including fifth and sixth standardized cumulants into (2) probably would add little in terms of accuracy. Instead, the formula will be more complicated and the practical intuition behind the formula will be lost. Second, to capture the bimodal shape of the loss distribution for their credit portfolio composed of a few large exposures Beran and Ocker (2003) introduce a 'recursive' SPA. This approximation consists of applying the standard SPA on a sample of the credit portfolio for

[16] For portfolios 2 and 3 we obtain similar results as for portfolio 1 whereas the results for portfolio 4 confirm those of portfolio 5. We refrain from reporting them to save space.

which the extreme exposures are excluded and afterwards adjusting the preliminary estimated distribution for the extreme exposures. However, although, the recursive approximation considerably improves the accuracy of the distribution estimates, the error often remains important. These two proposed solutions still use the normal distribution as the underlying distribution for the SPA. The third solution, proposed by Wood et al (1993), and Booth and Wood (1995) is to replace the normal distribution in the LR and the BN formulas by another distribution. For the approximation of tail probabilities of the first passage time of a random walk with drift process they propose the Inverse Gaussian as an alternative distribution. The modified (inverse Gaussian based) LR and BN SPAs prove to give highly accurate estimation results in the tails when the distribution is characterized by large higher moments. However, the modified (inverse Gaussian based) SPA can never totally replace the normal based SPA as the distribution estimates show to be less accurate when the higher standardized cumulants are low. Hence, one 'standard-to-fit-all' SPA does not exist.

5. Conclusions

CR+ is an influential model for portfolio credit risk, which provides analytical tools to derive a credit loss distribution. The model attracted a lot attention from the practitioners and scholars due to its analytical tractability which eliminates the need for Monte Carlo simulation. However, the original Panjer recursion of the CR+ model is unstable and therefore often yields inaccurate results for the tail of the loss distribution of credit portfolios. To overcome this problem several alternative solutions were proposed among which the SPA, which is shown to be extremely fast, more robust in practical applications and accurate for large portfolios regardless the complexity of the risk factor structure. However, it was argued in the literature that SPAs are not always exact and reliable and that sometimes its relative errors are quite large. In this paper we find that the accuracy of the SPA for CR+ strongly depends on how much the higher order standardized cumulants and the relative error (6) vary with θ. On the example of five real-life credit portfolios we show that the commonly applied SPAs (the LR and the BN formulas) are bad substitutes to the original Panjer recursion when the third and the fourth standardized cumulants of the loss distribution of the credit portfolio and the relative error calculated at $\theta = 0$ deviate significantly from their limits and vary strongly with θ.

We conclude that the accuracy and reliability of the LR and the BN formulas deteriorate when the third and the fourth moments of the loss distribution increase. Indeed, both formulas give accurate VaR estimates for portfolios with low third and fourth standardized cumulants, such as portfolio 1. For all other portfolios they produced VaR estimates beyond the 95% confidence interval. Moreover, larger skewness and kurtosis of credit loss distribution cause numerically instability of the LR formula. Therefore we warn against applying SPA without preliminary checking the higher standardized cumulants of the loss distribution.

At the same time we corroborate the results by Gordy (2002) that for the simple one risk factor CR+ specification, the original Panjer recursion algorithm is accurate and robust, even for highly heterogeneous credit portfolios with big skewness and kurtosis of the loss distribution.

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Appendix. The third and the fourth standardized cumulants in CR+.

In this section we show that the third and fourth standardized cumulants are dependent on θ. This proof is based on the derivations of cgf for CR+ made by Gordy (2002). In order to make the proof simpler we make the following assumptions:

1. There is only one systematic risk factor;
2. The credit portfolio is absolutely homogeneous, i.e. all default exposures are of equal size. In such simple portfolio only one exposure band is possible;
3. The standardized loss unit is equal to the size of the default exposure and thus in the event of default by obligor i, there is a fixed loss $\nu_i = 1$.

In this case the cgf for CR+ is

$$K(\theta) = \tau \log\left(\frac{\mu(1-\delta)}{\mu - \delta\, \Omega(\theta)} \right),$$

where $\Omega(\theta) = \sum_i w_i \bar{p}_i \exp(\nu_i \theta)$, $\mu = \sum_i w_i \bar{p}_i$ $\delta = \mu/(\tau + \mu)$ and $\tau = 1/\sigma$.

As in Gordy (2002) we denote D the differential operator, which means that $D^j f(x)$ is the jth derivative of f with respect to x. The jth derivative of $\Omega(\theta)$ is given by the following equation:

$$D^j \Omega(\theta) = \sum_i w_i \bar{p}_i \nu_i^j \exp(\nu_i \theta) \quad \forall j \geq 0$$

The first derivative of $\psi(\theta)$ is:

$$K'(\theta) = \tau \left(\frac{\delta D \Omega(\theta)}{\mu - \delta\, \Omega(\theta)} \right)$$

The expression in the parenthesis is generalised as

$$V_j(\theta) = \frac{\delta D^j \Omega(\theta)}{\mu - \delta\, \Omega(\theta)}$$

The derivatives of $K(\theta)$ can be generated recursively by

$$K'(\theta) = \tau V_1(\theta)$$
$$K''(\theta) = \tau(V_2(\theta) + V_1(\theta)^2)$$
$$K'''(\theta) = \tau(V_3(\theta) + 3V_2(\theta)V_1(\theta) + 2V_1(\theta)^3)$$
$$K''''(\theta) = \tau(V_4(\theta) + 4V_3(\theta)V_1(\theta) + 3V_2(\theta)^2 + 12V_2(\theta)V_1(\theta)^2 + 6V_1(\theta)^4)$$

If $\nu_i = 1$, then all derivatives $D^i\Omega(\theta)$ are equal to $\Omega(\theta)$, therefore

$$V_j(\theta) = \frac{\delta\,\Omega(\theta)}{\mu - \delta\,\Omega(\theta)} = V_{j+1}(\theta) = V(\theta).$$

Then in one risk factor case the third standardized cumulant is

$$\zeta_3(\theta) = \frac{\kappa_3(\theta)}{\kappa_2^{3/2}(\theta)} = \frac{K'''(\theta)}{(K''(\theta))^{3/2}} = \frac{\tau\left[V(\theta) + 3V^2(\theta) + 2V^3(\theta)\right]}{\left[\tau\left(V(\theta) + V^2(\theta)\right)\right]^{3/2}} =$$
$$= \frac{1 + 2V(\theta)}{\sqrt{\tau\left(V(\theta) + V(\theta)^2\right)}} = \frac{1}{\sqrt{\tau\left(V(\theta) + V^2(\theta)\right)}} + \frac{2}{\sqrt{\tau\left(\frac{1}{V(\theta)} + 1\right)}}$$

The fourth standardized cumulant is

$$\zeta_4(\theta) = \frac{\kappa_4(\theta)}{\kappa_2^2(\theta)} = \frac{K''''(\theta)}{(K''(\theta))^2} = \frac{\tau\left(V(\theta) + 7V(\theta)^2 + 12V(\theta)^3 + 6V(\theta)^4\right)}{\tau^2\left(V(\theta)^2 + 2V(\theta)^3 + V(\theta)^4\right)} =$$
$$= \frac{V(\theta) + V(\theta)^2}{\tau\left(V(\theta)^2 + 2V(\theta)^3 + V(\theta)^4\right)} + \frac{6}{\tau} = \frac{1}{\tau}\left(\frac{1}{V(\theta) + V(\theta)^2} + 6\right)$$

The relative error is

$$\left(\frac{\zeta_4(\theta)}{8} - \frac{5\zeta_3(\theta)^2}{24}\right) = -\frac{1}{12\tau\left(V(\theta) + V(\theta)^2\right)} - \frac{1}{12\tau}$$

Let $\hat{\theta}$ be the value at which the denominator in $V(\theta)$ is equal to zero. Therefore the denominator in $V(\theta)$ is positive and decreasing for all $\theta < \hat{\theta}$ and equal to zero at $\hat{\theta}$. The numerator in $V(\theta)$ is positive and increasing, so $V(\theta)$ is positive and increasing for all $\theta < \hat{\theta}$. As $\theta < \hat{\theta}$ then

$$V(\theta) \to \infty \quad \zeta_3(\theta) \to 2/\sqrt{\tau}, \quad \zeta_4(\theta) \to 6/\tau, \quad \left(\frac{\zeta_4(\theta)}{8} - \frac{5\zeta_3(\theta)^2}{24}\right) \to -\frac{1}{12\tau}$$

In case when $\nu_i > 1$, the dependency of the standardized cumulants and the correction term on θ is more complex, but as we show empirically the standardized cumulants and the correction term have the limits derived above.

Table 1. Summary statistics.

This table provides summary statistics for the five real-life credit portfolios randomly sampled from a large real-life credit portfolio. Each of the sampled portfolios consists of 1,000 obligors. The difference between the five portfolios is situated in the skewness and kurtosis of the loss distribution with minimal values for portfolio 1 and maximal for portfolio 5. We increase skewness and kurtosis by increasing maximum exposure to a single borrower, increasing the difference between the first and second largest exposures and keeping the number of exceptionally high exposures restricted to no more than four. In the first part of the table we report statistics of the exposure size distributions. Summary statistics of the credit loss distributions are reported in the second part of the table. The summary statistics of the credit loss distributions were calculated for one systematic risk factor CR+ specification with LGD=0.3 and σ^2=1.

	Portfolio 1	**Portfolio 2**	**Portfolio 3**	**Portfolio 4**	**Portfolio 5**
			Exposure size distribution		
Maximal exposure*	0.94%	1.50%	1.73%	4.35%	4.29%
Minimal exposure*	0.00%	0.00%	0.00%	0.00%	0.00%
Skewness	3.59	5.07	5.86	12.46	12.23
Kurtosis	19.34	37.02	50.30	248.28	198.98
			Credit loss distribution		
Expected Loss*	0.33%	0.04%	0.03%	0.03%	0.03%
St Deviation*	0.35%	0.06%	0.05%	0.05%	0.06%
Skewness**	2.03	3.14	4.37	4.71	5.68
Kurtosis**	9.11	18.15	36.34	60.66	81.27

* expressed in percentage to book value
** skewness and kurtosis were calculated without preliminary dividing portfolio into exposure bands and therefore differ from results reported in figure 3.

Table 2. Risk estimates and distribution characteristics of the credit portfolios.

In this table we report Monte Carlo, Panjer, LR and BN VaRs for the five portfolios and the probability that Monte Carlo loss will exceed these estimates. We show results only for VaR_{α} at 90%, 95%, 97.5%, 99% and 99.5% levels. VaR is expressed as a percentage of the total book value. The goodness-of-fit is estimated by the Kolmogorov-Smirnov test.

Portfolio 1	90	95	97.5	99	99.5	KS value
MC VaR‡	0.782	1.022	1.271	1.589	1.836	
Panjer VaR‡	0.783	1.027	1.271	1.593	1.837	0.026
Pr[MC loss > Panjer VaR] (%)	9.959	4.917	2.495	0.989	0.499	
LR VaR‡	0.779	1.022	1.264	1.586	1.828	0.026
Pr[MC loss > LR VaR] (%)	10.077	5.007	2.549	1.013	0.509	
BN VaR‡	0.782	1.025	1.268	1.590	1.833	0.026
Pr[MC loss > BN VaR] (%)	9.989	4.954	2.523	0.999	0.505	
Portfolio 2						
MC VaR‡	0.115	0.167	0.224	0.311	0.374	
Panjer VaR‡	0.112	0.166	0.223	0.308	0.374	0.051
Pr[MC loss > Panjer VaR] (%)	10.360***	5.100	2.527	1.038	0.500	
LR VaR‡	0.126	0.185	0.245	0.324	0.385	0.179
Pr[MC loss > LR VaR] (%)	8.498***	4.020***	1.999***	0.858***	0.460	
BN VaR‡	0.128	0.187	0.247	0.326	0.387	0.205
Pr[MC loss > BN VaR] (%)	8.244***	3.949***	1.958***	0.845***	0.454**	
Portfolio 3						
MC VaR‡	0.075	0.113	0.152	0.210	0.258	
Panjer VaR‡	0.072	0.112	0.152	0.211	0.264	0.051
Pr[MC loss > Panjer VaR] (%)	10.499***	5.088	2.514	0.987	0.462	
LR VaR‡	0.080	0.141	0.200	0.278	0.337	0.231
Pr[MC loss > LR VaR] (%)	9.065***	2.957***	1.173***	0.393***	0.224***	
BN VaR‡	0.086	0.144	0.202	0.280	0.339	0.256
Pr[MC loss > BN VaR] (%)	8.062***	2.800***	1.128***	0.389***	0.221***	
Portfolio 4						
MC VaR‡	0.087	0.127	0.173	0.257	0.305	
Panjer VaR‡	0.084	0.125	0.173	0.255	0.306	0.051
Pr[MC loss > Panjer VaR] (%)	10.479***	5.163**	2.501	1.023	0.485	
LR VaR‡	0.049	0.054	0.062	0.409	0.514	0.667***
Pr[MC loss > LR VaR] (%)	20.973***	18.952***	15.913***	0.114***	0.031***	
BN VaR‡	0.055	0.165	0.277	0.417	0.520	0.308***
Pr[MC loss > BN VaR] (%)	18.426***	2.760***	0.778***	0.104***	0.030***	
Portfolio 5						
MC VaR‡	0.089	0.129	0.181	0.262	0.320	
Panjer VaR‡	0.086	0.127	0.179	0.259	0.317	0.077
Pr[MC loss > Panjer VaR] (%)	10.630***	5.176**	2.555	1.045	0.509	
LR VaR‡	0.044	0.051	0.062	0.087	0.556	0.769***
Pr[MC loss > LR VaR] (%)	24.908***	21.507***	16.967***	10.425***	0.062***	
BN VaR‡	0.050	0.185	0.303	0.451	0.562	0.282*
Pr[MC loss > BN VaR] (%)	21.798***	2.387***	0.599***	0.184***	0.059***	

Asterisks indicate the level of significance as follows: (*) 0.10, (**) 0.05, (***) 0.01.
We use the following abbreviations: **MC** – Monte Carlo simulations; **LR** – the Lugannani-Rice formula; **BN** – the Barndorff-Nielsen formula; **KS** – Kolmogorov-Smirnov test.
‡ - reported as a percentage of the total book value.

Figure 1. Credit loss distributions

The histograms present simulated loss distributions of the five credit portfolios under the following assumptions: one systematic risk factor, LGD=0.3 and σ^2=1. In order to ease the comparison between the portfolios, loss is calculated as a percentage to the total book value.

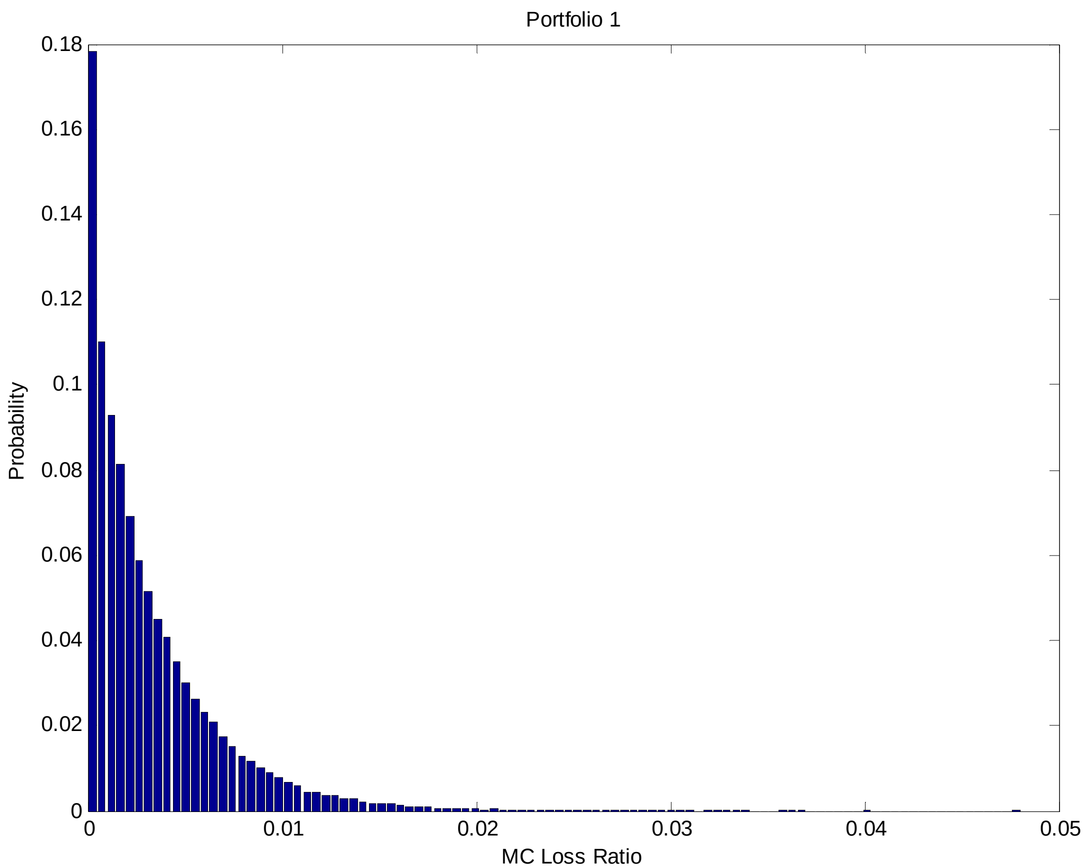


Figure 1. (continued)

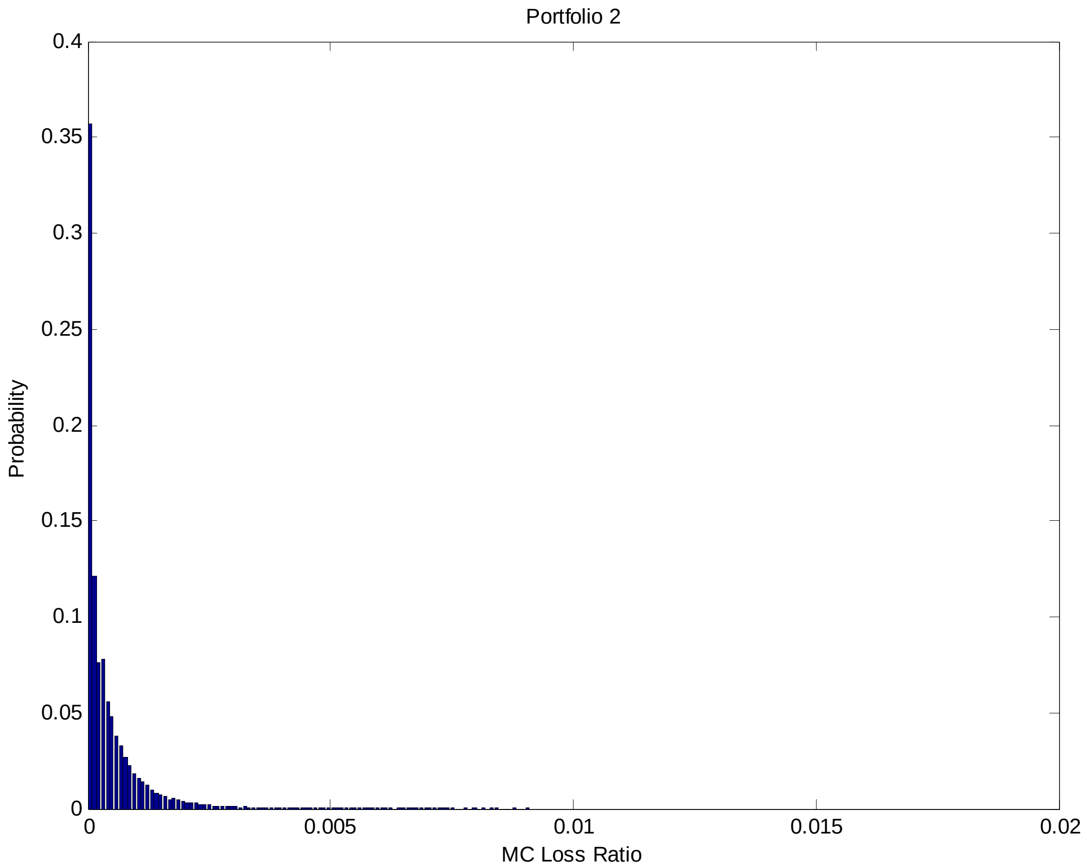


Figure 1. (continued)

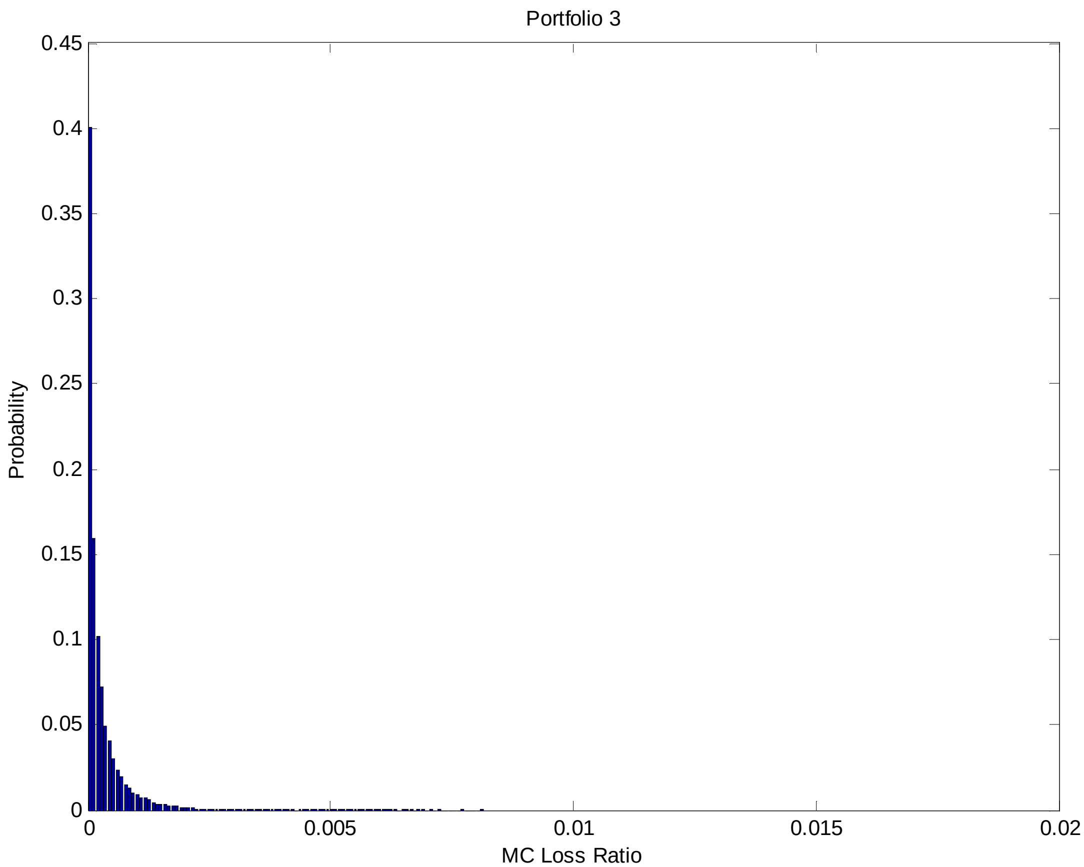


Figure 1. (continued)

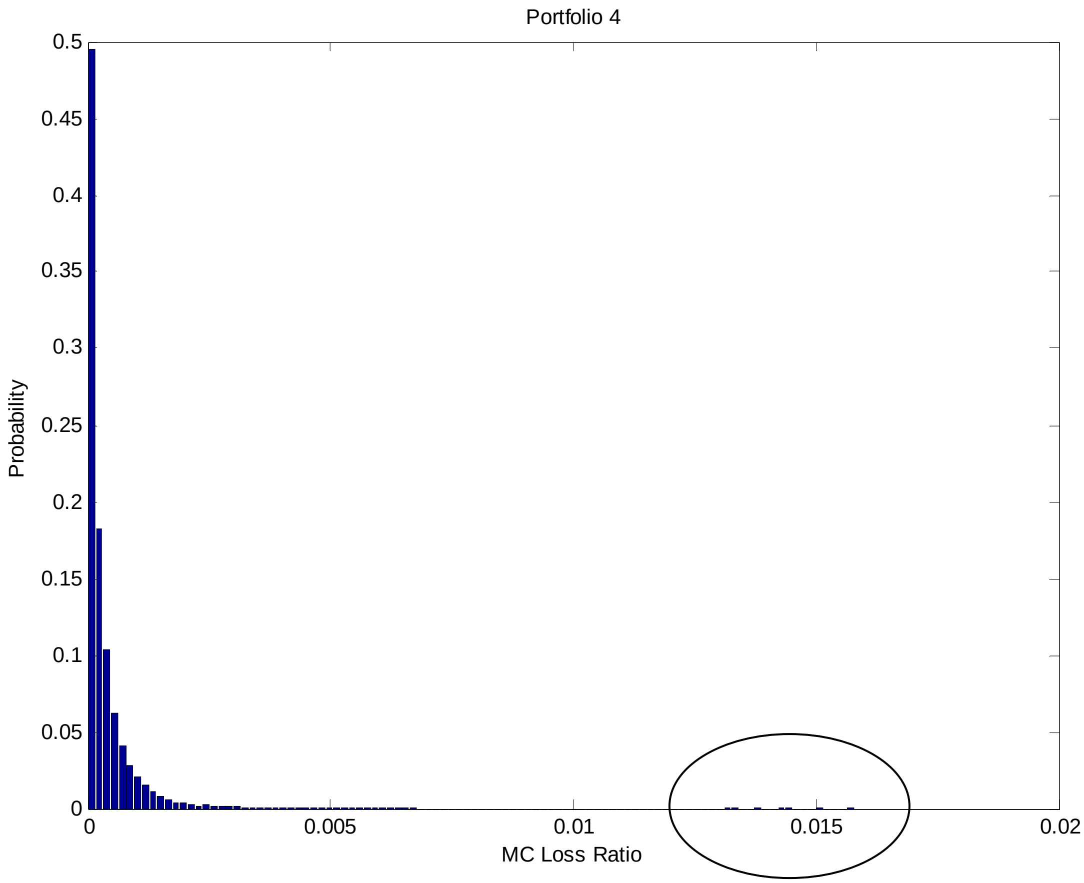


Figure 1. (continued)

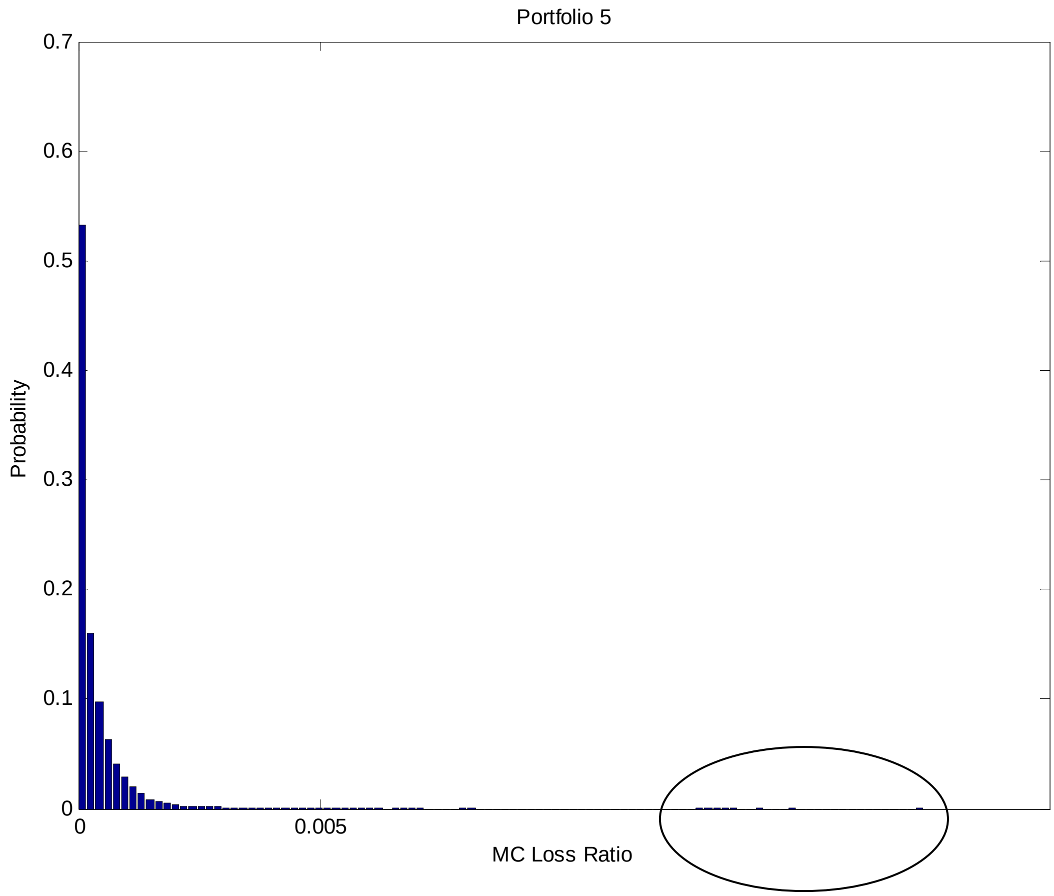
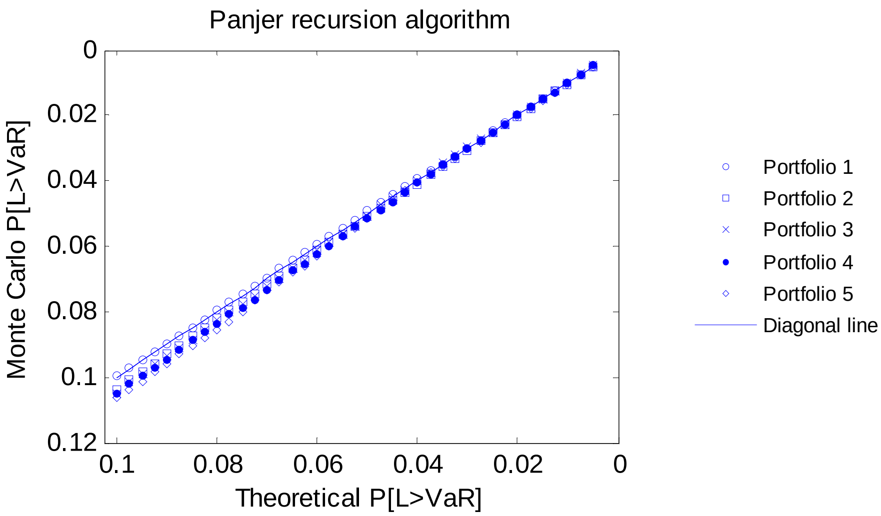


Figure 2. Q-Q plots.

This figure presents Q-Q plots where the probabilities of exceeding Panjer, LR and BN VaRs are plotted against the probabilities according to Monte Carlo simulations for all five portfolios. A good fit is indicated by values plotted on the diagonal (the full straight line), which means that the estimated probability of having a loss higher than VaR and the ‘true’ probability calculated via the Monte Carlo simulations are equal. If the values plot below the diagonal, then the Monte Carlo probability exceeds the estimated and VaR is underestimated; if it is above, then it is overestimated.



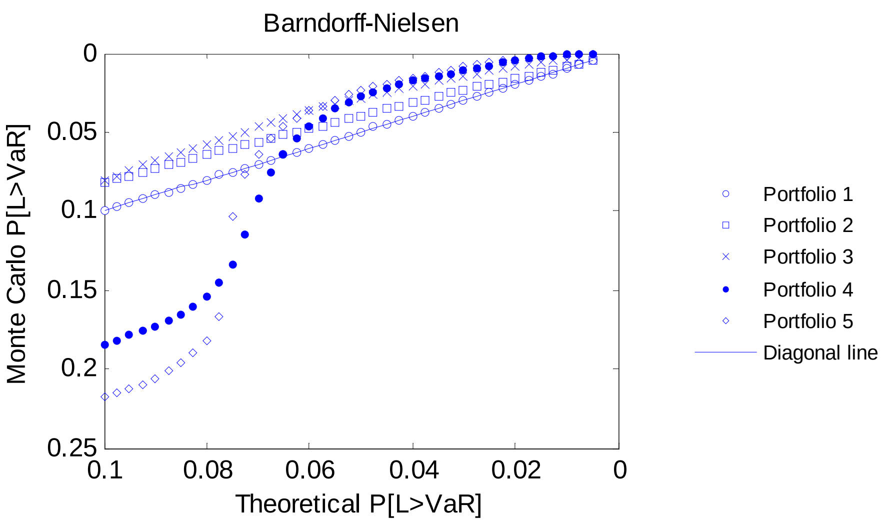


Figure 2. (continued)

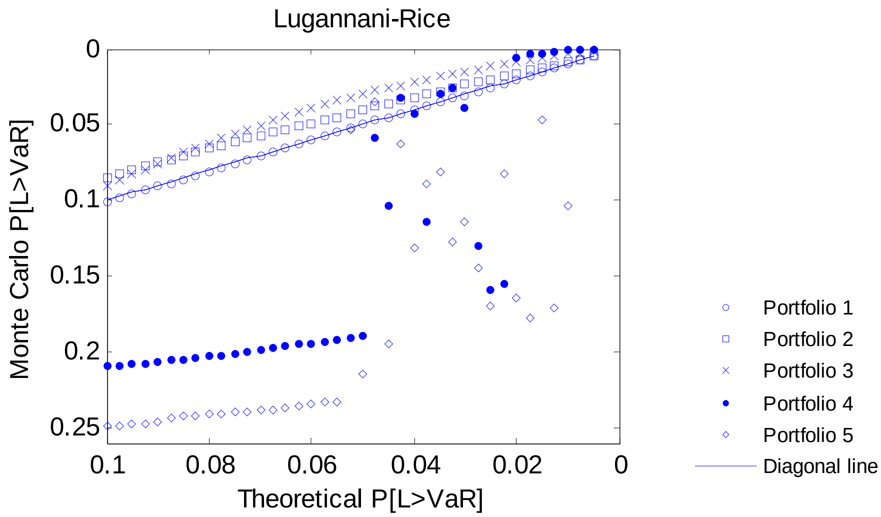


Figure 3. Dependence of $\zeta_3(\theta)$ and $\zeta_4(\theta)$ and the relative error $\left(\frac{\zeta_4(\theta)}{8} - \frac{5\zeta_3(\theta)^2}{24}\right)$ on θ.

In this figure the values of the third, the fourth standardized cumulants and the relative error are plotted against θ for one systematic risk factor CR+ model with LGD=0.3 and σ^2=1. At first, all exposures were divided in 100 exposure bands and then the standardized cumulants and the relative error were calculated.

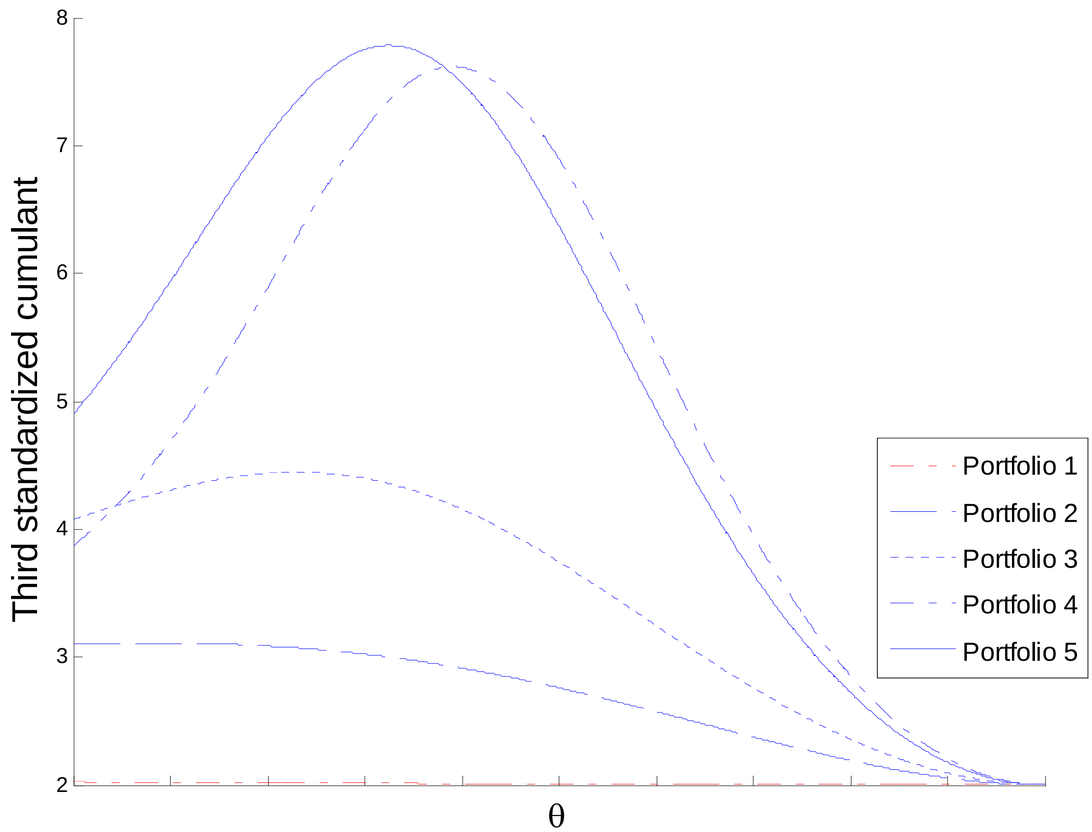


Figure 3. (continued)

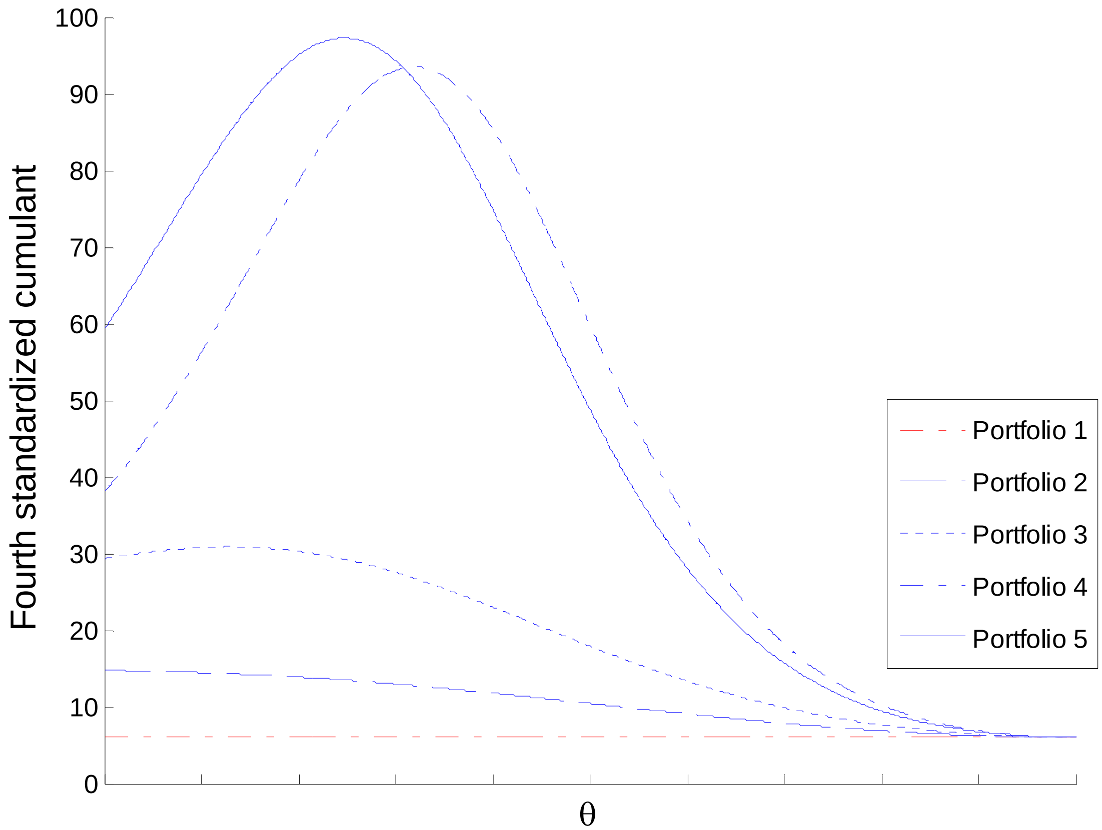


Figure 3. (continued)

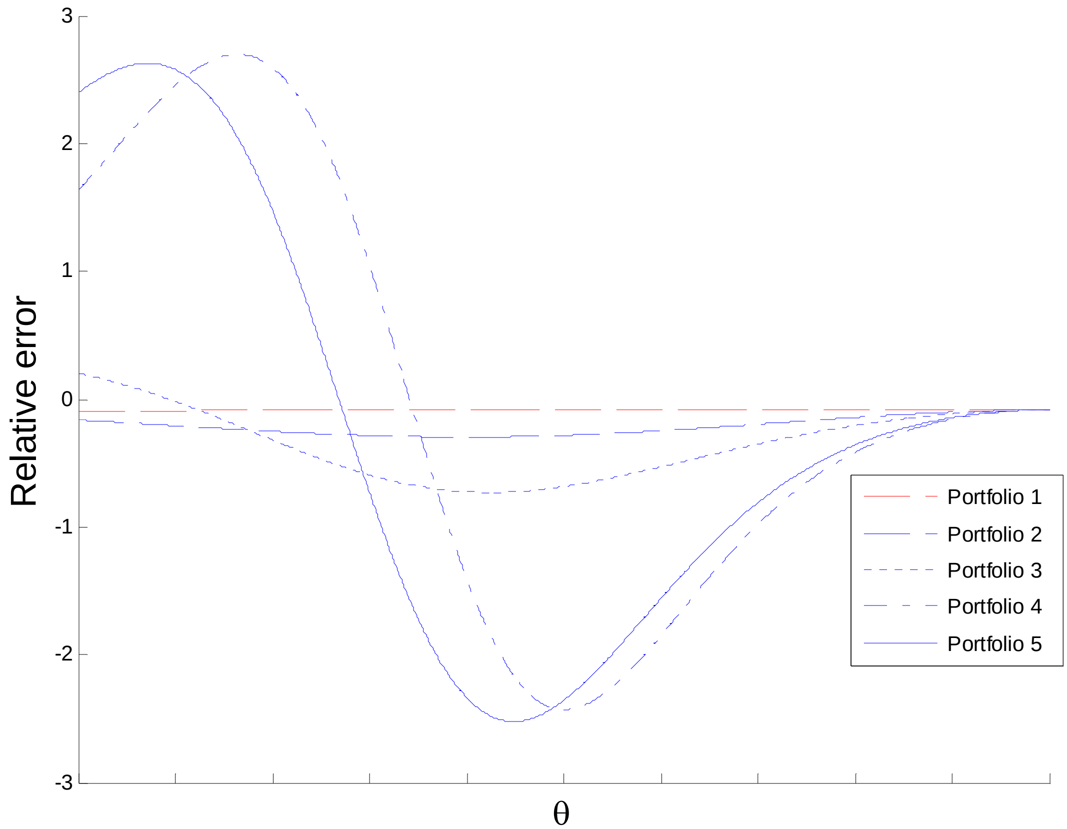


Figure 4. Loss exceedance curves.

This figure shows the loss exceedance curves produced by the LR formula for the first and the fifth credit portfolios, which are the extreme cases. In order to apply the LR formula a fine grid of 1,000 values in the open interval (0, $\hat{\theta}$) is formed and then at each point in the grid the pairs of losses and corresponding probabilities are computed. In order to ease the comparison between the portfolios, we do not report loss in pecuniary terms. Instead the probabilities are plotted against the points in the grids.

Increasing loss exceedance curve of portfolio 5 implies negative probability of losses. Loss exceeding curve of portfolio 4 was not reported here, although it also increases in some intervals. The loss exceeding curves of portfolios 2 and 3 are non-increasing functions of the loss value.